

# Partial Orders

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Introduction to Discrete Mathematics  
Sections 7.6 of Rosen  
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Consider the recent renovation of Avery Hall. In this process several things had to be done.

- Remove Asbestos
- Replace Windows
- Paint Walls
- Refinish Floors
- Assign Offices
- Move in Office-Furniture.

Clearly, some things had to be done before others could even begin—Asbestos had to be removed before *anything*; painting had to be done before the floors to avoid ruining them, etc.

On the other hand, several things could have been done concurrently—painting could be done while replacing the windows and assigning office could have been done at anytime.

Such a scenario can be nicely modeled using *partial orderings*.

**Definition**

A relation  $R$  on a set  $S$  is called a *partial order* if it is reflexive, antisymmetric and transitive. A set  $S$  together with a partial ordering  $R$  is called a *partially ordered set* or *poset* for short and is denoted

$$(S, R)$$

Partial orderings are used to give an order to sets that may not have a natural one. In our renovation example, we could define an ordering such that  $(a, b) \in R$  if  $a$  *must* be done before  $b$  can be done.

We use the notation

$$a \preceq b$$

to indicate that  $(a, b) \in R$  is a partial order and

$$a \prec b$$

when  $a \neq b$ .

The notation  $\prec$  is not to be mistaken for “less than equal to.”  
Rather,  $\prec$  is used to denote *any* partial ordering.

Latex notation: `\preccurlyeq`, `\prec`.

## Definition

The elements  $a$  and  $b$  of a poset  $(S, \preceq)$  are called comparable if either  $a \preceq b$  or  $b \preceq a$ . When  $a, b \in S$  such that neither are comparable, we say that they are *incomparable*.

Looking back at our renovation example, we can see that

Remove Asbestos  $\prec a_i$

for all activities  $a_i$ . Also,

Paint Walls  $\prec$  Refinish Floors

Some items are also incomparable—replacing windows can be done before, after or during the assignment of offices.

### Definition

If  $(S, \preceq)$  is a poset and every two elements of  $S$  are comparable,  $S$  is called a *totally ordered set*. The relation  $\preceq$  is said to be a *total order*.

### Example

The set of integers over the relation “less than equal to” is a total order;  $(\mathbb{Z}, \leq)$  since for every  $a, b \in \mathbb{Z}$ , it must be the case that  $a \leq b$  or  $b \leq a$ .

What happens if we replace  $\leq$  with  $<$ ?

### Definition

$(S, \preceq)$  is a *well-ordered set* if it is a poset such that  $\preceq$  is a total ordering and such that every nonempty subset of  $S$  has a *least element*

### Example

The natural numbers along with  $\leq$ ,  $(\mathbb{N}, \leq)$  is a well-ordered set since any subset of  $\mathbb{N}$  will have a least element and  $\leq$  is a total ordering on  $\mathbb{N}$  as before.

However,  $(\mathbb{Z}, \leq)$  is not a well-ordered set. Why? Is it totally ordered?



Well-ordered sets are the basis of the proof technique known as *induction* (more when we cover Chapter 3).

### Theorem (Principle of Well-Ordered Induction)

*Suppose that  $S$  is a well ordered set. Then  $P(x)$  is true for all  $x \in S$  if*

**Basis Step:**  *$P(x_0)$  is true for the least element of  $S$  and*

**Induction Step:** *For every  $y \in S$  if  $P(x)$  is true for all  $x \prec y$  then  $P(y)$  is true.*

Suppose it is not the case that  $(P(x))$  holds for all  $x \in S \Rightarrow \exists y P(y)$  is false  $\Rightarrow A = \{x \in S \mid P(x) \text{ is false}\}$  is not empty.

Since  $S$  is well ordered,  $A$  has a least element  $a$ .

$P(x_0)$  is true  $\Rightarrow a \neq x_0$ .

$P(x)$  holds for all  $x \in S$  and  $x \prec a$ , then  $P(a)$  holds, by the induction step.

This yields a contradiction. □

Lexicographic ordering is the same as any dictionary or phone book—we use alphabetical order starting with the first character in the string, then the next character (if the first was equal) etc. (you can consider “no character” for shorter words to be less than “a”).

Formally, lexicographic ordering is defined by combining two other orderings.

### Definition

Let  $(A_1, \preceq_1)$  and  $(A_2, \preceq_2)$  be two posets. The *lexicographic ordering*  $\preceq$  on the Cartesian product  $A_1 \times A_2$  is defined by

$$(a_1, a_2) \preceq (a'_1, a'_2)$$

if  $a_1 \prec_1 a'_1$  or if  $a_1 = a'_1$  and  $a_2 \preceq_2 a'_2$ .

Lexicographic ordering generalizes to the Cartesian product of  $n$  sets in the natural way.

Define  $\preceq$  on  $A_1 \times A_2 \times \cdots \times A_n$  by

$$(a_1, a_2, \dots, a_n) \prec (b_1, b_2, \dots, b_n)$$

if  $a_1 \prec b_1$  or if there is an integer  $i > 0$  such that

$$a_1 = b_1, a_2 = b_2, \dots, a_i = b_i$$

and  $a_{i+1} \prec b_{i+1}$

Consider the two non-equal strings  $a_1a_2 \cdots a_m$  and  $b_1b_2 \cdots b_n$  on a poset  $S$ .

Let  $t = \min(n, m)$  and  $\prec$  is the lexicographic ordering on  $S^t$ .

$a_1a_2 \cdots a_m$  is less than  $b_1b_2 \cdots b_n$  if and only if

- $(a_1, a_2, \dots, a_t) \prec (b_1, b_2, \dots, b_t)$ , or
- $(a_1, a_2, \dots, a_t) = (b_1, b_2, \dots, b_t)$  and  $m < n$

As with relations and functions, there is a convenient graphical representation for partial orders—*Hasse Diagrams*.

Consider the digraph representation of a partial order—since we *know* we are dealing with a partial order, we *implicitly* know that the relation must be reflexive and transitive. Thus we can simplify the graph as follows:

- Remove all self-loops.
- Remove all transitive edges.
- Make the graph direction-less—that is, we can assume that the orientations are *upwards*.

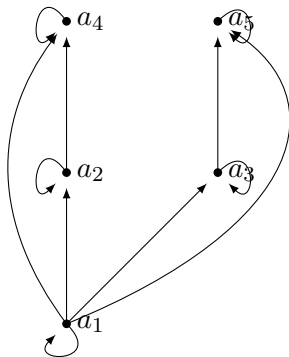
The resulting diagram is far simpler.

# Hasse Diagram

## Example

Partial Orders

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Remove Self-Loops

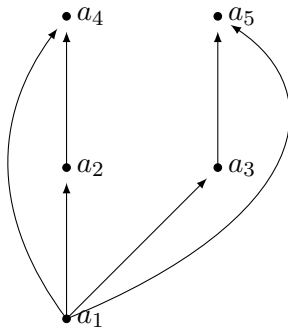


# Hasse Diagram

## Example

Partial Orders

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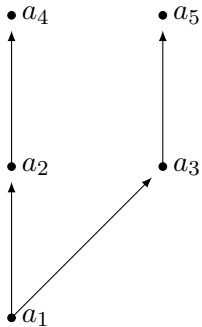
Remove Transitive Loops

# Hasse Diagram

## Example

Partial Orders

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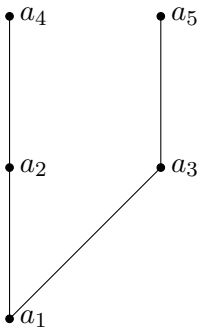
Remove Orientation

# Hasse Diagram

## Example

Partial Orders

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Hasse Diagram!

# Hasse Diagrams

## Example

Of course, you need not always start with the complete relation in the partial order and then trim everything. Rather, you can build a Hasse directly from the partial order.

### Example

Draw a Hasse diagram for the partial ordering

$$\{(a, b) \mid a \mid b\}$$

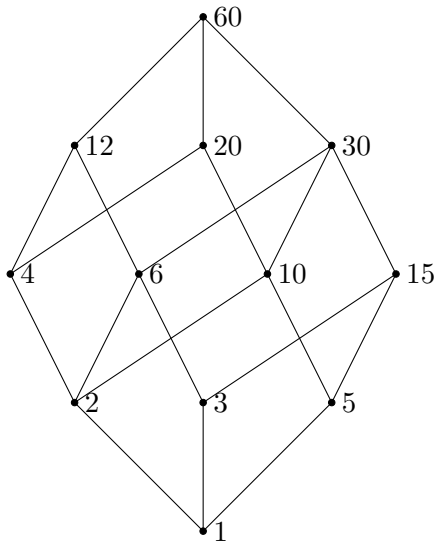
on  $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$  (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)

# Hasse Diagrams

Example Answer

Partial Orders

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We will define the following terms:

- A maximal/minimal element in a poset  $(S, \preceq)$ .
- The maximum (greatest)/minimum (least) element of a poset  $(S, \preceq)$ .
- An upper/lower bound element of a subset  $A$  of a poset  $(S, \preceq)$ .
- The greatest upper/least lower bound element of a subset  $A$  of a poset  $(S, \preceq)$ .
- Lattice

### Definition

An element  $a$  in a poset  $(S, \preceq)$  is called *maximal* if it is not less than any other element in  $S$ . That is,

$$\nexists b \in S (a \prec b)$$

If there is one *unique* maximal element  $a$ , we call it the *maximum* element (or the *greatest element*).

### Definition

An element  $a$  in a poset  $(S, \preceq)$  is called *minimal* if it is not greater than any other element in  $S$ . That is,

$$\nexists b \in S (b \prec a)$$

If there is one *unique* minimal element  $a$ , we call it the *minimum* element (or the *least element*).



### Definition

Let  $(S, \preceq)$  be a poset and let  $A \subseteq S$ . If  $u$  is an element of  $S$  such that  $a \preceq u$  for all elements  $a \in A$  then  $u$  is an *upper bound* of  $A$ .

An element  $x$  that is an upper bound on a subset  $A$  and is less than all other upper bounds on  $A$  is called the *least upper bound* on  $A$ . We abbreviate “lub”.

### Definition

Let  $(S, \preceq)$  be a poset and let  $A \subseteq S$ . If  $l$  is an element of  $S$  such that  $l \preceq a$  for all elements  $a \in A$  then  $l$  is a *lower bound* of  $A$ .

An element  $x$  that is a lower bound on a subset  $A$  and is greater than all other lower bounds on  $A$  is called the *greatest lower bound* on  $A$ . We abbreviate “glb”.

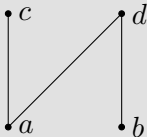
# Extremal Elements

## Example I

Partial Orders

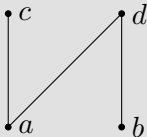
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### Example



What are the minimal, maximal, minimum, maximum elements?

## Example



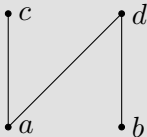
What are the minimal, maximal, minimum, maximum elements?

- Minimal:  $\{a, b\}$

# Extremal Elements

## Example I

### Example



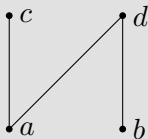
What are the minimal, maximal, minimum, maximum elements?

- Minimal:  $\{a, b\}$
- Maximal:  $\{c, d\}$

# Extremal Elements

## Example I

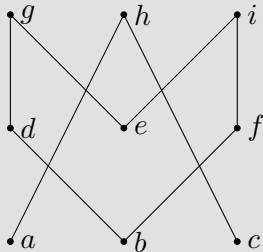
### Example



What are the minimal, maximal, minimum, maximum elements?

- Minimal:  $\{a, b\}$
- Maximal:  $\{c, d\}$
- There are no unique minimal or maximal elements.

## Example



What are the lower/upper bounds and glb/lub of the sets  $\{d, e, f\}$ ,  $\{a, c\}$  and  $\{b, d\}$

# Extremal Elements

## Example II

Partial Orders

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 $\{d, e, f\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.

 $\{a, c\}$  $\{b, d\}$



# Extremal Elements

## Example II

Partial Orders

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 $\{d, e, f\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\emptyset$ , thus no lub either.

 $\{a, c\}$  $\{b, d\}$

# Extremal Elements

## Example II

Partial Orders

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 $\{d, e, f\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\emptyset$ , thus no lub either.

 $\{a, c\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.

 $\{b, d\}$

# Extremal Elements

## Example II

Partial Orders

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 $\{d, e, f\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\emptyset$ , thus no lub either.

 $\{a, c\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\{h\}$ , since its unique, lub is also  $h$ .

 $\{b, d\}$

$\{d, e, f\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\emptyset$ , thus no lub either.

 $\{a, c\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\{h\}$ , since its unique, lub is also  $h$ .

 $\{b, d\}$ 

- Lower Bounds:  $\{b\}$  and so also glb.

$\{d, e, f\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\emptyset$ , thus no lub either.

 $\{a, c\}$ 

- Lower Bounds:  $\emptyset$ , thus no glb either.
- Upper Bounds:  $\{h\}$ , since its unique, lub is also  $h$ .

 $\{b, d\}$ 

- Lower Bounds:  $\{b\}$  and so also glb.
- Upper Bounds:  $\{d, g\}$  and since  $d \prec g$ , the lub is  $d$ .

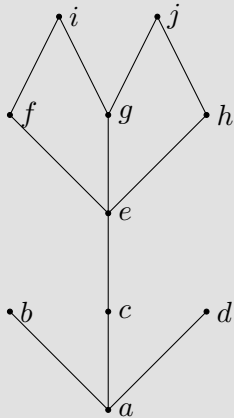
# Extremal Elements

## Example III

Partial Orders

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### Example



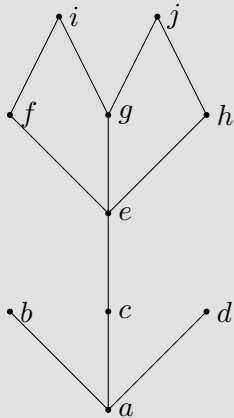
Minimal/Maximal elements?

Bounds, glb, lub of  $\{c, e\}$ ?Bounds, glb, lub of  $\{b, i\}$ ?

# Extremal Elements

## Example III

### Example



Minimal/Maximal elements?

- Minimal & Minimum Element:  $a$ .

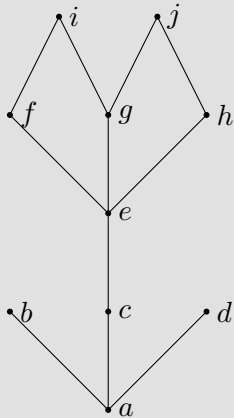
Bounds, glb, lub of  $\{c, e\}$ ?

Bounds, glb, lub of  $\{b, i\}$ ?

# Extremal Elements

## Example III

### Example



Minimal/Maximal elements?

- Minimal & Minimum Element:  $a$ .
- Maximal Elements:  $b, d, i, j$ .

Bounds, glb, lub of  $\{c, e\}$ ?

Bounds, glb, lub of  $\{b, i\}$ ?



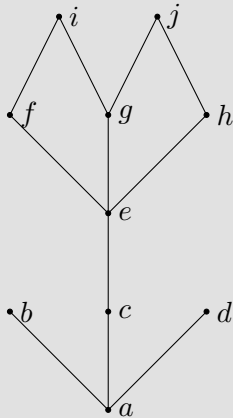
# Extremal Elements

## Example III

Partial Orders

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### Example



Minimal/Maximal elements?

- Minimal & Minimum Element:  $a$ .
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Bounds, glb, lub of  $\{c, e\}$ ?

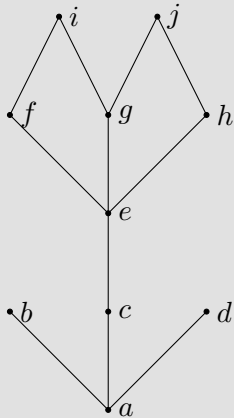
- Lower Bounds:  $\{a, c\}$ , thus glb is  $c$ .

Bounds, glb, lub of  $\{b, i\}$ ?

# Extremal Elements

## Example III

### Example



Minimal/Maximal elements?

- Minimal & Minimum Element:  $a$ .
- Maximal Elements:  $b, d, i, j$ .

Bounds, glb, lub of  $\{c, e\}$ ?

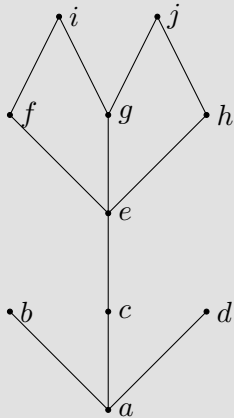
- Lower Bounds:  $\{a, c\}$ , thus glb is  $c$ .
- Upper Bounds:  $\{e, f, g, h, i, j\}$  thus lub is  $e$

Bounds, glb, lub of  $\{b, i\}$ ?

# Extremal Elements

## Example III

### Example



Minimal/Maximal elements?

- Minimal & Minimum Element:  $a$ .
- Maximal Elements:  $b, d, i, j$ .

Bounds, glb, lub of  $\{c, e\}$ ?

- Lower Bounds:  $\{a, c\}$ , thus glb is  $c$ .
- Upper Bounds:  $\{e, f, g, h, i, j\}$  thus lub is  $e$ .

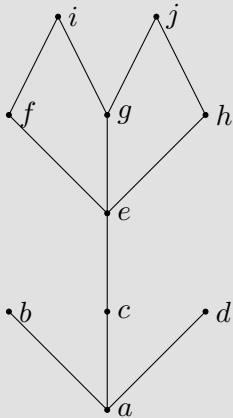
Bounds, glb, lub of  $\{b, i\}$ ?

- Lower Bounds:  $\{a\}$ , thus glb is  $a$ .

# Extremal Elements

## Example III

### Example



Minimal/Maximal elements?

- Minimal & Minimum Element:  $a$ .
- Maximal Elements:  $b, d, i, j$ .

Bounds, glb, lub of  $\{c, e\}$ ?

- Lower Bounds:  $\{a, c\}$ , thus glb is  $c$ .
- Upper Bounds:  $\{e, f, g, h, i, j\}$  thus lub is  $e$ .

Bounds, glb, lub of  $\{b, i\}$ ?

- Lower Bounds:  $\{a\}$ , thus glb is  $a$ .
- Upper Bounds:  $\emptyset$ , thus lub DNE.

A special structure arises when every pair of elements in a poset has a lub and glb.

### Definition

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

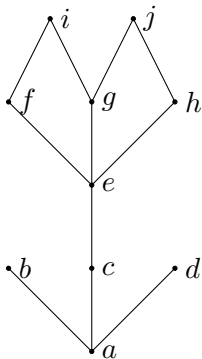
# Lattices

## Example

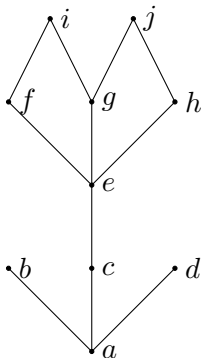
Partial Orders

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Is the example from before a lattice?



Is the example from before a lattice?



No, since the pair  $(b, c)$  do not have a least upper bound.

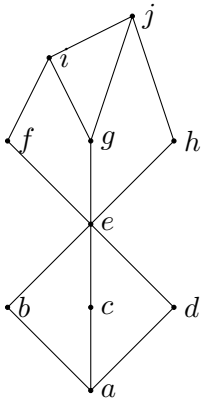
# Lattices

## Example

Partial Orders

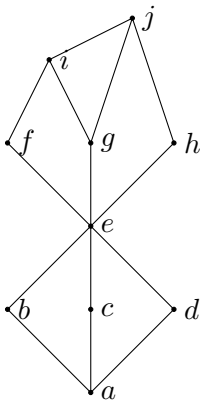
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What if we modified it as follows?





What if we modified it as follows?



Yes, it is now a lattice, since for any pair, there is a lub & glb.

To show that a partial order is not a lattice, it suffices to find a pair that does not have a lub/glb.

For a pair not to have a lub/glb, they must first be *incomparable*. (Why?)

You can then view the upper/lower bounds on a pair as a sub-hasse diagram; if there is no *minimum* element in this sub-diagram, then it is not a lattice.

Let us return to the introductory example of the Avery renovation. Now that we have got a partial order model, it would be nice to actually create a concrete schedule.

That is, given a partial order, we would like to transform it into a *total order* that is *compatible* with the partial order.

A total order is compatible if it doesn't violate any of the original relations in the partial ordering.

Essentially, we are simply imposing an order on incomparable elements in the partial order.

Before we give the algorithm, we need some tools to justify its correctness.

### Fact

*Every finite, nonempty poset  $(S, \preceq)$  has a minimal element.*

We will prove by a form of reductio ad absurdum.

**Proof.**

Assume to the contrary that a nonempty, finite (WLOG, assume  $|S| = n$ ) poset  $(S, \preceq)$  has no minimal element. In particular,  $a_1$  is not a minimal element.



**Proof.**

Assume to the contrary that a nonempty, finite (WLOG, assume  $|S| = n$ ) poset  $(S, \preceq)$  has no minimal element. In particular,  $a_1$  is not a minimal element.

If  $a_1$  is not minimal, then there exists  $a_2$  such that  $a_2 \prec a_1$ . But also,  $a_2$  is not minimal by the assumption.



## Proof.

Assume to the contrary that a nonempty, finite (WLOG, assume  $|S| = n$ ) poset  $(S, \preceq)$  has no minimal element. In particular,  $a_1$  is not a minimal element.

If  $a_1$  is not minimal, then there exists  $a_2$  such that  $a_2 \prec a_1$ . But also,  $a_2$  is not minimal by the assumption.

Therefore, there exists  $a_3$  such that  $a_3 \prec a_2$ . This process proceeds until we have the last element,  $a_n$  thus,

$$a_n \prec a_{n-1} \prec \cdots a_2 \prec a_1$$

thus by definition  $a_n$  is the minimal element. □

The idea to topological sorting is that we start with a poset  $(S, \preceq)$  and remove a minimal element (choosing arbitrarily if there are more than one). Such an element is guaranteed to exist by the previous fact.

As we remove each minimal element, the set shrinks. Thus, we are guaranteed the algorithm will halt in a finite number of steps.

Furthermore, the order in which elements are removed is a total order;

$$a_1 \prec a_2 \prec \cdots \prec a_n$$

We now present the algorithm itself.



## TOPOLOGICAL SORT

INPUT :  $(S, \preceq)$  a poset with  $|S| = n$

OUTPUT : A total ordering  $(a_1, a_2, \dots, a_n)$

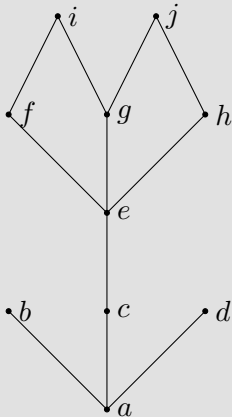
```
1  $k = 1$ 
2 WHILE  $S \neq \emptyset$  DO
3      $a_k \leftarrow$  a minimal element in  $S$ 
4      $S = S \setminus \{a_k\}$ 
5      $k = k + 1$ 
6 END
7 return  $(a_1, a_2, \dots, a_n)$ 
```

# Topological Sorting

## Example

### Example

Find a compatible ordering (topological ordering) of the poset represented by the diagram below.



Questions?