

Discrete Mathematics

Trees

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1 Trees

- Introduction
- Rooted Trees
- Binary Trees
- Decision Trees

2 Tree Problems

- Minimum Spanning Tree

1 Trees

- Introduction
- Rooted Trees
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2 Tree Problems

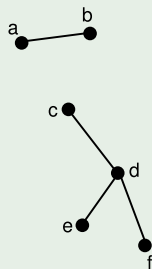
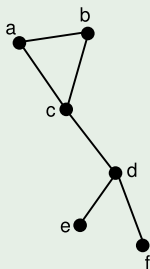
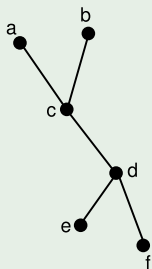
- Minimum Spanning Tree

Tree

Definition

tree: connected graph with no cycle

examples



Tree Theorems

Theorem

T is a tree (T is connected and contains no cycle).

\Leftrightarrow

*There is one and only one path
between any two distinct nodes in T .*

\Leftrightarrow

*T is connected, but if any edge is removed
it will no longer be connected.*

\Leftrightarrow

*T contains no cycle, but if an edge is added
between any pair of nodes one and only one cycle will be formed.*

Theorem

$$|E| = |V| - 1$$

- proof method: induction on the number of edges

Tree Theorems

Proof: Base step.

- $|E| = 0 \Rightarrow |V| = 1$
- $|E| = 1 \Rightarrow |V| = 2$
- $|E| = 2 \Rightarrow |V| = 3$
- assume that $|E| = |V| - 1$ for $|E| \leq k$

Tree Theorems

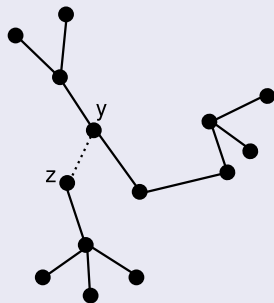
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Tree Theorems

Proof: Induction step.

- $|E| = k + 1$



- remove edge (y, z) :
 $T_1 = (V_1, E_1), T_2 = (V_2, E_2)$

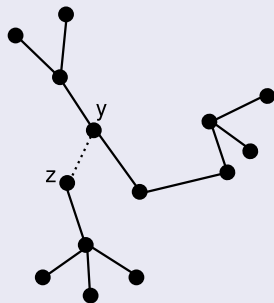
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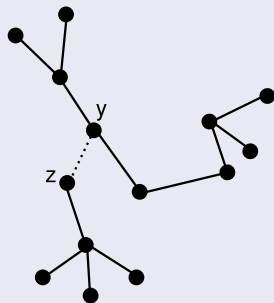
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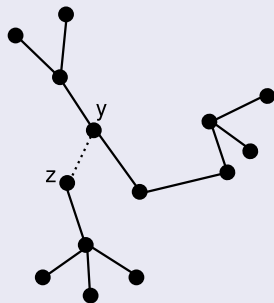
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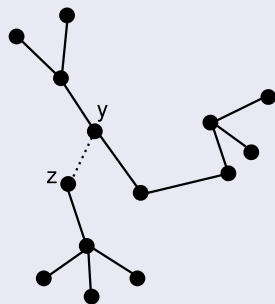
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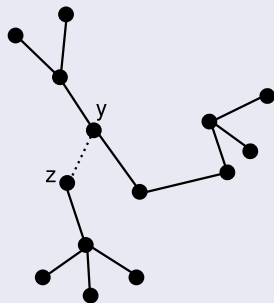
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Theorem

T is a tree (T is connected and contains no cycle).

\Leftrightarrow

T is connected and $|E| = |V| - 1$.

\Leftrightarrow

T contains no cycle and $|E| = |V| - 1$.

Tree Theorems

Theorem

In a tree, there are at least two nodes with degree 1.

Proof.

- $2|E| = \sum_{v \in V} d_v$
- assume: only 1 node with degree 1:
 - $\Rightarrow 2|E| \geq 2(|V| - 1) + 1$
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Rooted Tree

- hierarchy between nodes
- creates implicit direction on edges: in and out degrees
- in-degree 0: **root** (only 1 such node)
- out-degree 0: **leaf**
- not a leaf: **internal** node

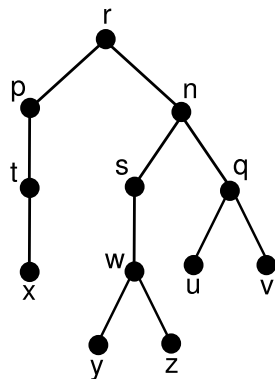
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Node Level

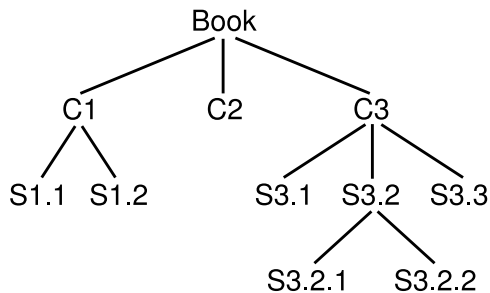
- **level** of node: distance from root
- *parent*: adjacent node closer to root (only 1 such node)
- *child*: adjacent nodes further from root
- *sibling*: nodes with same parent
- **depth** of tree: maximum level in tree

Rooted Tree Example



- root: r
- leaves: $x y z u v$
- internal nodes: $r p n t s q w$
- parent of y : w
children of w : y and z
- y and z are siblings

Rooted Tree Example



Book

- C1
 - S1.1
 - S1.2
- C2
- C3
 - S3.1
 - S3.2
 - S3.2.1
 - S3.2.2
 - S3.3

Ordered Rooted Tree

- siblings ordered from left to right
- **universal address system**
- root: 0
- children of root: 1, 2, 3, ...
- v : internal node with address a
children of v : $a.1, a.2, a.3, \dots$

Lexicographic Order

- address A comes before address B if one of:

- $A = x_1x_2 \dots x_ix_j \dots$

- $B = x_1x_2 \dots x_ix_k \dots$

- x_j comes before x_k

- $A = x_1x_2 \dots x_i$

- $B = x_1x_2 \dots x_ix_k \dots$

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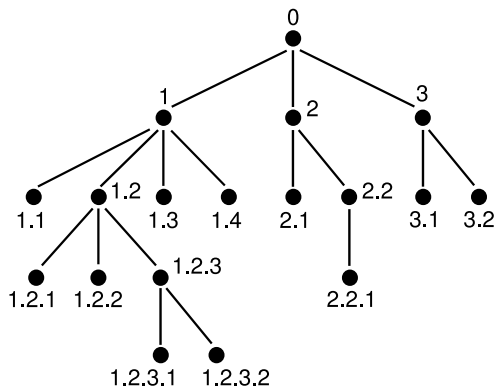
$$B = x_1x_2 \dots x_ix_k \dots$$

x_j comes before x_k

- $A = x_1x_2 \dots x_i$

$$B = x_1x_2 \dots x_ix_k \dots$$

Lexicographic Order Example



- 0 - 1 - 1.1 - 1.2
- 1.2.1 - 1.2.2 - 1.2.3
- 1.2.3.1 - 1.2.3.2
- 1.3 - 1.4 - 2
- 2.1 - 2.2 - 2.2.1
- 3 - 3.1 - 3.2

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Binary Trees

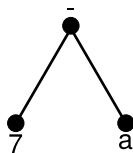
- $T = (V, E)$ is a **binary tree**:
 $\forall v \in V [d_v^o \in \{0, 1, 2\}]$
- $T = (V, E)$ is a *complete* binary tree:
 $\forall v \in V [d_v^o \in \{0, 2\}]$

Expression Tree

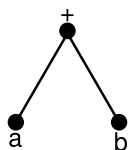
- binary operations can be represented as binary trees
- root: operator, children: operands
- mathematical expression can be represented as trees
- internal nodes: operators, leaves: variables and values

Expression Tree Examples

$7 - a$

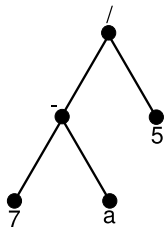


$a + b$

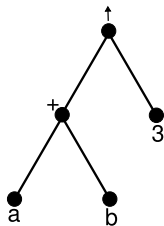


Expression Tree Examples

$$\frac{7 - a}{5}$$

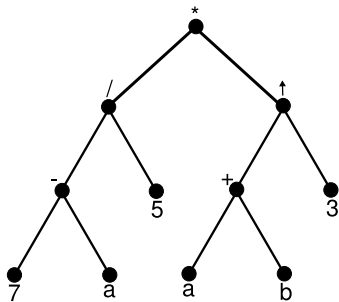


$$(a + b)^3$$



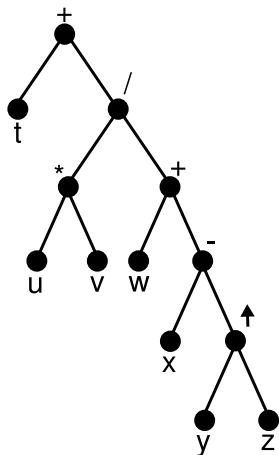
Expression Tree Examples

$$\frac{7 - a}{5} \cdot (a + b)^3$$



Expression Tree Examples

$$t + \frac{u * v}{w + x - yz}$$



Expression Tree Traversals

- 1 **inorder** traversal:
traverse left subtree, visit root, traverse right subtree
- 2 **preorder** traversal:
visit root, traverse left subtree, traverse right subtree
- 3 **postorder** traversal (reverse Polish notation):
traverse left subtree, traverse right subtree, visit root

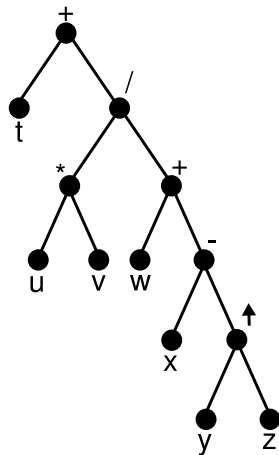
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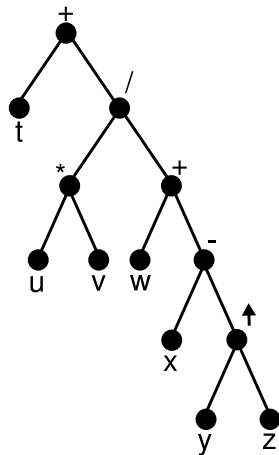
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Inorder Traversal Example



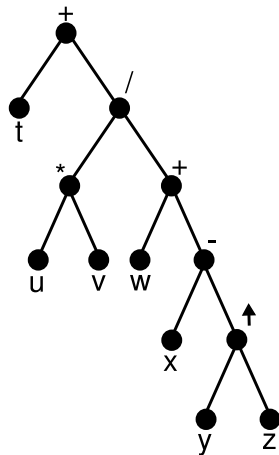
$t + u * v / w + x - y \uparrow z$

Preorder Traversal Example



$+ t / * u v + w - x \uparrow y z$

Postorder Traversal Example



$t u v * w x y z \uparrow - + / +$

Expression Tree Evaluation

- inorder traversal requires parentheses for precedence
- preorder and postorder traversals do not require parentheses

Postorder Evaluation Example

$t u v * w x y z \uparrow - + / +$
 $4 2 3 * 1 9 2 3 \uparrow - + / +$

4 2 3 *
4 6 1 9 2 3 \uparrow
4 6 1 9 8 -
4 6 1 1 +
4 6 2 /
4 3 +
7

Postorder Evaluation Example

$t\ u\ v\ * \ w\ x\ y\ z\ \uparrow\ -\ +\ /\ +$
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Regular Trees

- $T = (V, E)$ is an **m-ary tree**:
 $\forall v \in V [d_v^o \leq m]$
- $T = (V, E)$ is a complete m-ary tree:
 $\forall v \in V [d_v^o \in \{0, m\}]$

Regular Tree Theorem

Theorem

$T = (V, E)$: complete m -ary tree

- n : number of nodes
 - l : number of leaves
 - i : number of internal nodes
-
- $n = m \cdot i + 1$
 - $l = n - i = m \cdot i + 1 - i = (m - 1) \cdot i + 1$

$$i = \frac{l - 1}{m - 1}$$

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Regular Tree Examples

- how many matches are played in a tennis tournament with 27 players?
- every player is a leaf: $l = 27$
- every match is an internal node: $m = 2$
- number of matches: $i = \frac{l-1}{m-1} = \frac{27-1}{2-1} = 26$

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Regular Tree Examples

- how many extension cords with 4 outlets are required to connect 25 computers to a wall socket?
- every computer is a leaf: $l = 25$
- every extension cord is an internal node: $m = 4$
- number of cords: $i = \frac{l-1}{m-1} = \frac{25-1}{4-1} = 8$

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- Binary Trees
- Decision Trees

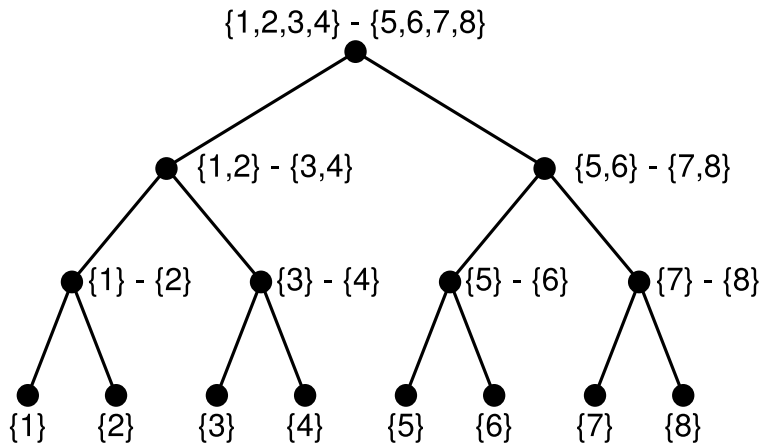
2 Tree Problems

- Minimum Spanning Tree

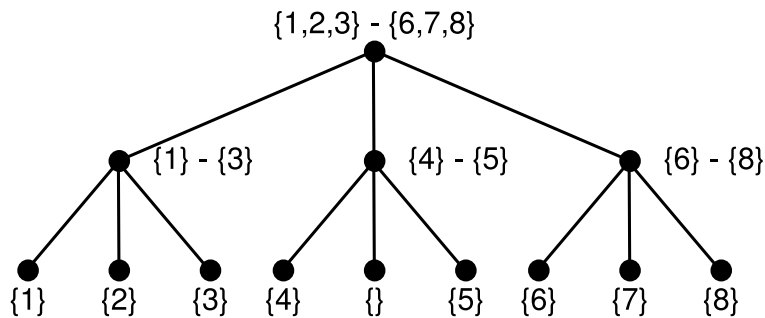
Decision Trees

- one of 8 coins is counterfeit (heavier)
- find counterfeit coin using a beam balance
- depth of tree: number of weighings

Decision Trees



Decision Trees



1 Trees

- Introduction
- Rooted Trees
- Binary Trees
- Decision Trees

2 Tree Problems

- Minimum Spanning Tree

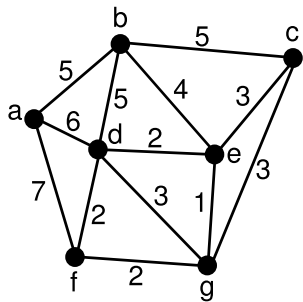
Spanning Tree

- $T = (V', E')$ is a **spanning tree** of $G(V, E)$:
 - T is a subgraph of G
 - T is a tree
 - $V' = V$
- **minimum spanning tree**:
 - total weight of edges in E' is minimal

Kruskal's Algorithm

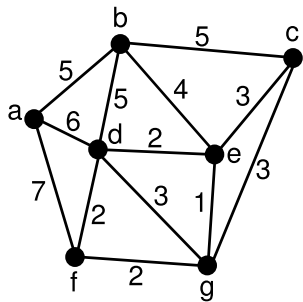
- 1 $G' = (V', E')$, $V' = \emptyset$, $E' = \emptyset$
- 2 select $e = (v_1, v_2) \in E - E'$ such that:
 $E' \cup \{e\}$ contains no cycle, and $wt(e)$ is minimal
- 3 $E' = E' \cup \{e\}$, $V' = V' \cup \{v_1, v_2\}$
- 4 if $|E'| = |V| - 1$: result is G'
- 5 go to step 2

Kruskal's Algorithm Example



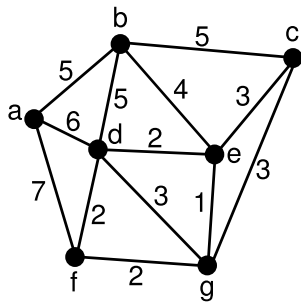
- minimum weight: 1
(e, g)
- $E' = \{(e, g)\}$
- $|E'| = 1$

Kruskal's Algorithm Example



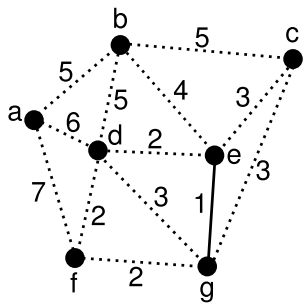
- minimum weight: 1
(e, g)
- $E' = \{(e, g)\}$
- $|E'| = 1$

Kruskal's Algorithm Example



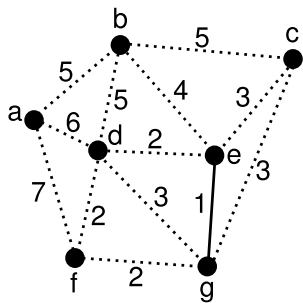
- minimum weight: 1
(e, g)
- $E' = \{(e, g)\}$
- $|E'| = 1$

Kruskal's Algorithm Example



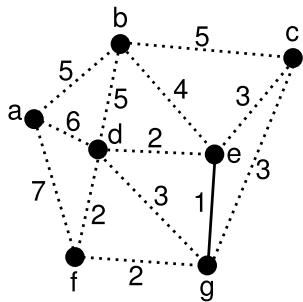
- minimum weight: 2
 $(d, e), (d, f), (f, g)$
- $E' = \{(e, g), (d, f)\}$
- $|E'| = 2$

Kruskal's Algorithm Example



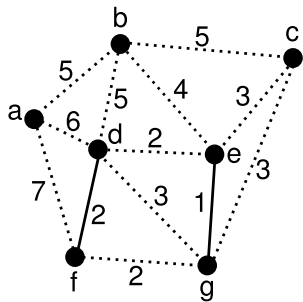
- minimum weight: 2
 $(d, e), (d, f), (f, g)$
- $E' = \{(e, g), (d, f)\}$
- $|E'| = 2$

Kruskal's Algorithm Example



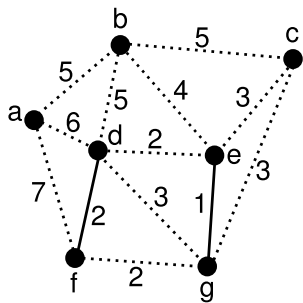
- minimum weight: 2
 $(d, e), (d, f), (f, g)$
- $E' = \{(e, g), (d, f)\}$
- $|E'| = 2$

Kruskal's Algorithm Example



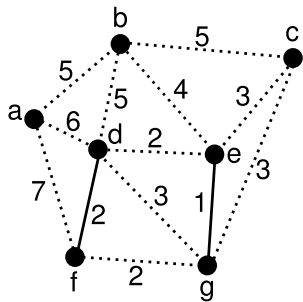
- minimum weight: 2
 $(d, e), (f, g)$
- $E' = \{(e, g), (d, f), (d, e)\}$
- $|E'| = 3$

Kruskal's Algorithm Example



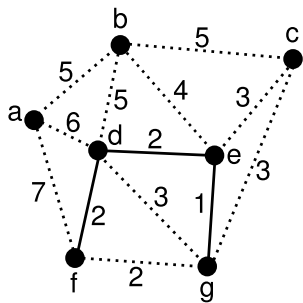
- minimum weight: 2
 $(d, e), (f, g)$
- $E' = \{(e, g), (d, f), (d, e)\}$
- $|E'| = 3$

Kruskal's Algorithm Example



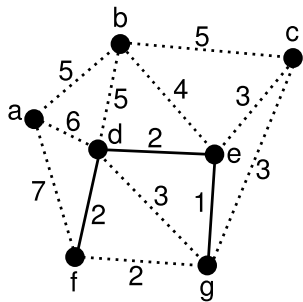
- minimum weight: 2
 $(d, e), (f, g)$
- $E' = \{(e, g), (d, f), (d, e)\}$
- $|E'| = 3$

Kruskal's Algorithm Example



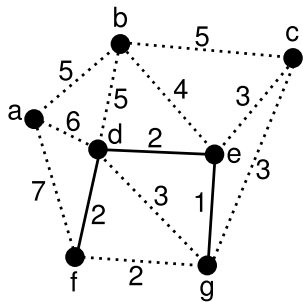
- minimum weight: 2
(f, g) forms a cycle
- minimum weight: 3
(c, e), (c, g), (d, g)
(d, g) forms a cycle
- $E' = \{(e, g), (d, f), (d, e), (c, e)\}$
- $|E'| = 4$

Kruskal's Algorithm Example



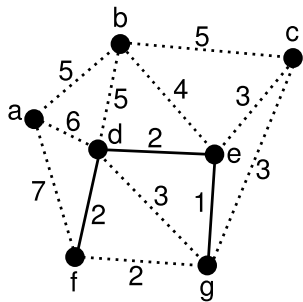
- minimum weight: 2
 (f, g) forms a cycle
- minimum weight: 3
 $(c, e), (c, g), (d, g)$
 (d, g) forms a cycle
- $E' = \{(e, g), (d, f), (d, e), (c, e)\}$
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Kruskal's Algorithm Example



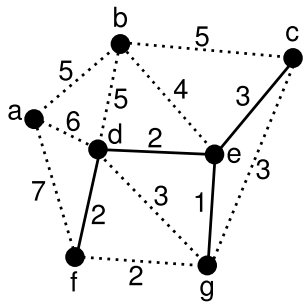
- minimum weight: 2
(f, g) forms a cycle
- minimum weight: 3
(c, e), (c, g), (d, g)
(d, g) forms a cycle
- $E' = \{(e, g), (d, f), (d, e), (c, e)\}$
- $|E'| = 4$

Kruskal's Algorithm Example



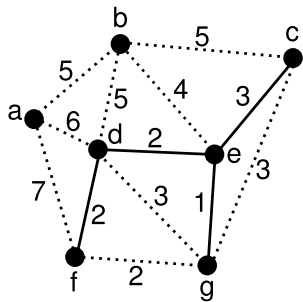
- minimum weight: 2
(f, g) forms a cycle
- minimum weight: 3
(c, e), (c, g), (d, g)
(d, g) forms a cycle
- $E' = \{(e, g), (d, f), (d, e), (c, e)\}$
- $|E'| = 4$

Kruskal's Algorithm Example



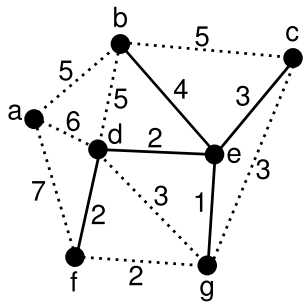
- $E' = \{$
 $(e, g), (d, f), (d, e),$
 $(c, e), (b, e)$
 $\}$
- $|E'| = 5$

Kruskal's Algorithm Example



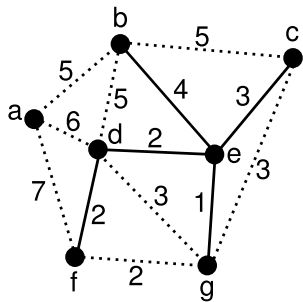
- $E' = \{$
 $(e, g), (d, f), (d, e),$
 $(c, e), (b, e)$
 $\}$
- $|E'| = 5$

Kruskal's Algorithm Example



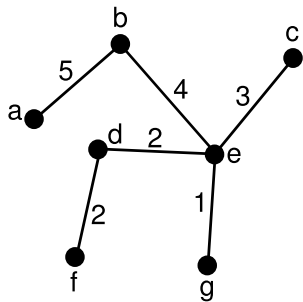
- $E' = \{$
 $(e, g), (d, f), (d, e),$
 $(c, e), (b, e), (a, b)$
 $\}$
- $|E'| = 6$

Kruskal's Algorithm Example



- $E' = \{$
 $(e, g), (d, f), (d, e),$
 $(c, e), (b, e), (a, b)$
 $\}$
- $|E'| = 6$

Kruskal's Algorithm Example

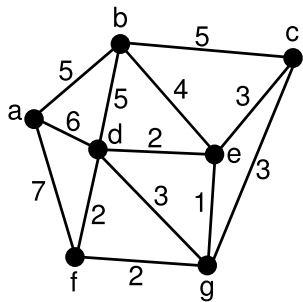


■ total weight: 17

Prim's Algorithm

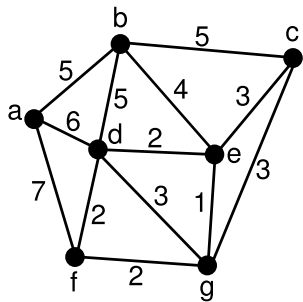
- 1 $T' = (V', E'), E' = \emptyset, v_0 \in V, V' = \{v_0\}$
- 2 select $v \in V - V'$ such that for a node $x \in V'$
 $e = (x, v), E' \cup \{e\}$ contains no cycle, and $wt(e)$ is minimal
- 3 $E' = E' \cup \{e\}, V' = V' \cup \{x\}$
- 4 if $|V'| = |V|$: result is T'
- 5 go to step 2

Prim's Algorithm Example



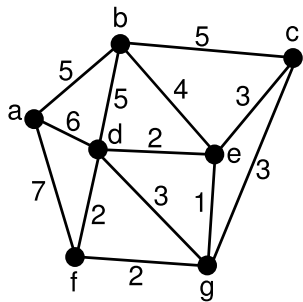
- $E' = \emptyset$
- $V' = \{a\}$
- $|V'| = 1$

Prim's Algorithm Example



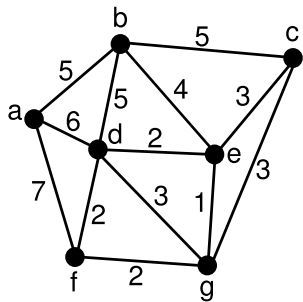
- $E' = \emptyset$
- $V' = \{a\}$
- $|V'| = 1$

Prim's Algorithm Example



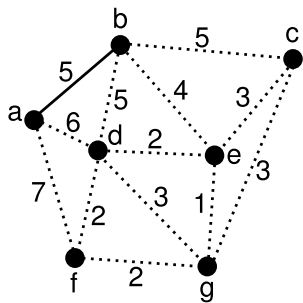
- $E' = \{(a, b)\}$
- $V' = \{a, b\}$
- $|V'| = 2$

Prim's Algorithm Example



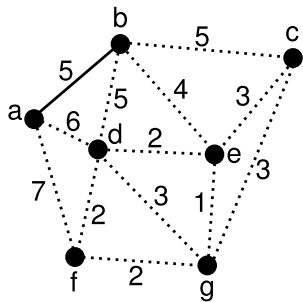
- $E' = \{(a, b)\}$
- $V' = \{a, b\}$
- $|V'| = 2$

Prim's Algorithm Example



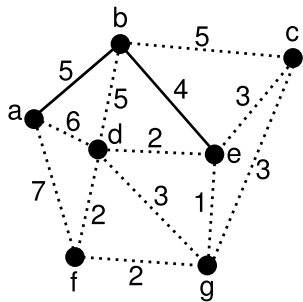
- $E' = \{(a, b), (b, e)\}$
- $V' = \{a, b, e\}$
- $|V'| = 3$

Prim's Algorithm Example



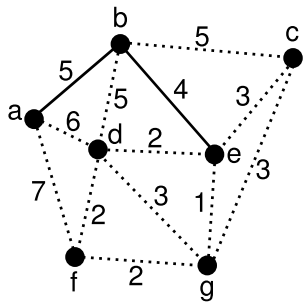
- $E' = \{(a, b), (b, e)\}$
- $V' = \{a, b, e\}$
- $|V'| = 3$

Prim's Algorithm Example



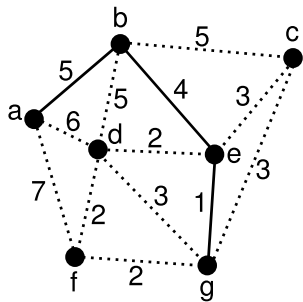
- $E' = \{(a, b), (b, e), (e, g)\}$
- $V' = \{a, b, e, g\}$
- $|V'| = 4$

Prim's Algorithm Example



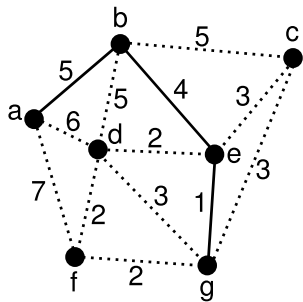
- $E' = \{(a, b), (b, e), (e, g)\}$
- $V' = \{a, b, e, g\}$
- $|V'| = 4$

Prim's Algorithm Example



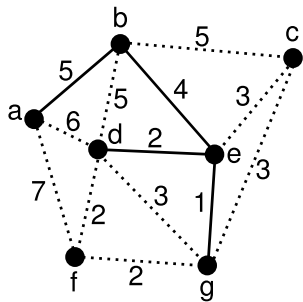
- $E' = \{(a, b), (b, e), (e, g), (d, e)\}$
- $V' = \{a, b, e, g, d\}$
- $|V'| = 5$

Prim's Algorithm Example



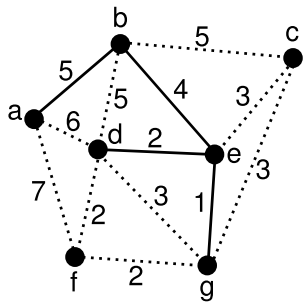
- $E' = \{(a, b), (b, e), (e, g), (d, e)\}$
- $V' = \{a, b, e, g, d\}$
- $|V'| = 5$

Prim's Algorithm Example



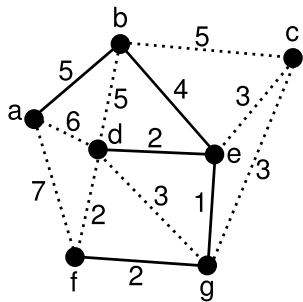
- $E' = \{ (a, b), (b, e), (e, g), (d, e), (f, g) \}$
- $V' = \{a, b, e, g, d, f\}$
- $|V'| = 6$

Prim's Algorithm Example



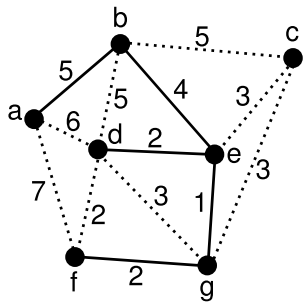
- $E' = \{$
 $(a, b), (b, e), (e, g),$
 $(d, e), (f, g)$
 $\}$
- $V' = \{a, b, e, g, d, f\}$
- $|V'| = 6$

Prim's Algorithm Example



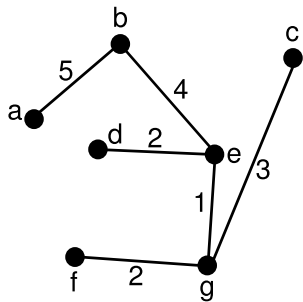
- $E' = \{$
 $(a, b), (b, e), (e, g),$
 $(d, e), (f, g), (c, g)$
 $\}$
- $V' = \{a, b, e, g, d, f, c\}$
- $|V'| = 7$

Prim's Algorithm Example



- $E' = \{$
 $(a, b), (b, e), (e, g),$
 $(d, e), (f, g), (c, g)$
 $\}$
- $V' = \{a, b, e, g, d, f, c\}$
- $|V'| = 7$

Prim's Algorithm Example



■ total weight: 17

Required Reading: Grimaldi

- Chapter 12: Trees
 - 12.1. Definitions and Examples
 - 12.2. Rooted Trees
- Chapter 13: Optimization and Matching
 - 13.2. Minimal Spanning Trees:
The Algorithms of Kruskal and Prim