Lists, higher order procedures, and symbols

6.037 - Structure and Interpretation of Computer Programs

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Lecture 2

Administrivia

- Project 0 was due today
- Reminder: Project 1 due at 7pm on Tuesday
- Mail to `6.037-psets@mit.edu`
- If you didn't sign up on Tuesday, let us know

Types

```
(+ 5 10)        =>
```

Types

```
(+ 5 10)        => 15
```

Types

```
(+ 5 10)        => 15

(+ "hi" 15)     =>
```

Types

```
(+ 5 10)      => 15

(+ "hi" 15)   =>
   +: expects type <number> as 1st argument,
      given: "hi"; other arguments were: 15
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Types

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- Addition is not defined for strings

Types

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(+ 5 10)      => 15

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   +: expects type <number> as 1st argument,
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```

- Addition is not defined for strings
- Only works for things of type number
- Scheme checks types for simple built-in functions

Simple data types

Everything has a type:

- Number

Simple data types

Everything has a type:

- Number
- String

Simple data types

Everything has a type:

- Number
- String
- Boolean

Simple data types

Everything has a type:

- Number
- String
- Boolean
- Procedures?

Simple data types

Everything has a type:

- Number
- String
- Boolean
- Procedures?
  - Is the type of `not` the same type as + ?

What about procedures?

- Procedures have their own types, based on arguments and return value
- **number** $\mapsto$ **number** means “takes one number, returns a number”

Type examples

```
(+ 5 10)      => 15

(+ "hi" 15)   =>
   +: expects type <number> as 1st argument,
      given: "hi"; other arguments were: 15
```

- What is the type of +?

Type examples

```
(+ 5 10)      => 15

(+ "hi" 15)   =>
   +: expects type <number> as 1st argument,
      given: "hi"; other arguments were: 15
```

- What is the type of $+$?
- **number, number $\mapsto$ number**

Type examples

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(+ 5 10)      => 15

(+ "hi" 15)   =>
   +: expects type <number> as 1st argument,
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- What is the type of $+$?
- **number, number** $\mapsto$ **number**

  (mostly)

Type examples

Expression:	…is of type:
`15`	
`"hi"`	
`square`	
`>`	

Type examples

Expression:	. . . is of type:
`15`	**number**
`"hi"`	
`square`	
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Type examples

Expression:	... is of type:
`15`	**number**
`"hi"`	**string**
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Type examples

Expression:	...is of type:
`15`	**number**
`"hi"`	**string**
`square`	**number** $\mapsto$ **number**
`>`	**number, number** $\mapsto$ **boolean**

- Type of a procedure is a contract
- If the operands have the specified types, the procedure will result in a value of the specified type
- Otherwise, its behavior is undefined

More complicated examples

```
(lambda (a b c)
  (if (> a 0) (+ b c) (- b c)))
```

, , $\mapsto$

More complicated examples

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number, number, number $\mapsto$ number

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number, number, number $\mapsto$ number

```
(lambda (p)
  (if p "hi" "bye"))
```

$\mapsto$

More complicated examples

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number, number, number $\mapsto$ **number**

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boolean $\mapsto$ **string**

```
(lambda (x)
 (* 3.14 (* 2 5)))
```

$\mapsto$

More complicated examples

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(lambda (a b c)
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any $\mapsto$

More complicated examples

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(lambda (a b c)
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number, number, number $\mapsto$ number

```
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boolean $\mapsto$ string

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(lambda (x)
 (* 3.14 (* 2 5)))
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any $\mapsto$ number

Patterns across procedures

Procedural abstraction is finding patterns, and making procedures of them:

- `(* 17 17)`
- `(* 42 42)`
- `(* x x)`
- ...

Patterns across procedures

Procedural abstraction is finding patterns, and making procedures of them:

- `(* 17 17)`
- `(* 42 42)`
- `(* x x)`
- ...
- `(lambda (x) (* x x))`

Summation

- $1 + 2 + \ldots + 100$
- $1 + 4 + 9 + \ldots + 100^2$
- $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{99^2} \approx \frac{\pi^2}{8}$

Summation

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(define (sum term a next b)
  (if (> a b) 0
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         (sum term (next a) next b))))
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Complex types

```
(define (sum term a next b)
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What is the type of this procedure?

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- What type is the output?

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$\mapsto$ **number**

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- What type is the output?
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Higher-order procedures take a procedure as an argument, or return one as a value

Higher-order procedures

$$\sum_{k=a}^{b} k$$

```
(define (sum-integers a b)
  (if (> a b) 0
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Higher-order procedures

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(define (new-sum-integers a b)
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```

Higher-order procedures

$$\sum_{\substack{k=a \\ k \text{ odd}}}^{b} \frac{1}{k^2} \approx \frac{\pi^2}{8}$$

```
(define (pi-sum a b)
  (if (> a b) 0
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(define (new-pi-sum a b)
  (sum (lambda (x) (/ 1 (square x)))
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       (lambda (x) (+ x 2))
       b))
```

Returning procedures

…takes a procedure as an argument or returns one as a value

```
(define (new-sum-integers a b)
  (sum (lambda (x) x) a (lambda (x) (+ x 1)) b))
```

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…takes a procedure as an argument or returns one as a value

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(define (add1 x) (+ x 1))
```

Returning procedures

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(define (new-sum-squares a b)
  (sum square a (lambda (x) (+ x 1)) b))
(define (add1 x) (+ x 1))
(define (new-sum-squares a b) (sum square a add1 b))
```

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(define (add1 x) (+ x 1))
(define (new-sum-squares a b) (sum square a add1 b))

(define (new-pi-sum a b)
  (sum (lambda (x) (/ 1 (square x))) a
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(define (new-sum-squares a b) (sum square a add1 b))

(define (new-pi-sum a b)
  (sum (lambda (x) (/ 1 (square x))) a
       (lambda (x) (+ x 2)) b))
```

Returning procedures

…takes a procedure as an argument or returns one as a value

```
(define (new-sum-integers a b)
  (sum (lambda (x) x) a (lambda (x) (+ x 1)) b))
(define (new-sum-squares a b)
  (sum square a (lambda (x) (+ x 1)) b))
(define (add1 x) (+ x 1))
(define (new-sum-squares a b) (sum square a add1 b))

(define (new-pi-sum a b)
  (sum (lambda (x) (/ 1 (square x))) a
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(define (add2 x) (+ x 2))
```

Returning procedures

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```

Returning procedures

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(define (add1 x) (+ x 1))
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Returning procedures

```
(define (add1 x) (+ x 1))
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(define incrementby (lambda (n) ... ))
```

Returning procedures

```
(define (add1 x) (+ x 1))
(define (add2 x) (+ x 2))

(define incrementby (lambda (n) ... ))

(define add1 (incrementby 1))
```

Returning procedures

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(define (add1 x) (+ x 1))
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(define add1 (incrementby 1))
(define add2 (incrementby 2))
```

Returning procedures

```
(define (add1 x) (+ x 1))
(define (add2 x) (+ x 2))

(define incrementby (lambda (n) ... ))

(define add1 (incrementby 1))
(define add2 (incrementby 2))
(define add37.5 (incrementby 37.5))
```

Returning procedures

```
(define (add1 x) (+ x 1))
(define (add2 x) (+ x 2))

(define incrementby (lambda (n) ... ))

(define add1 (incrementby 1))
(define add2 (incrementby 2))
(define add37.5 (incrementby 37.5))
```

type of `incrementby`:

Returning procedures

```
(define (add1 x) (+ x 1))
(define (add2 x) (+ x 2))

(define incrementby (lambda (n) ... ))

(define add1 (incrementby 1))
(define add2 (incrementby 2))
(define add37.5 (incrementby 37.5))
```

type of `incrementby`:
number $\mapsto$ (number $\mapsto$ number)

Returning procedures

```
(define incrementby
  ; type: num -> (num->num)
  (lambda (n)                              ))
```

Returning procedures

```
(define incrementby
  ; type: num -> (num->num)
  (lambda (n) (lambda (x) (+ x n))))
```

Returning procedures

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(define incrementby
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```

Returning procedures

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(  incrementby                          2 )
( (lambda (n) (lambda (x) (+ x n))) 2 )
```

Returning procedures

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(define incrementby
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              (lambda (x) (+ x 2))
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Returning procedures

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  (lambda (n) (lambda (x) (+ x n))))

(  incrementby                        2 )
( (lambda (n) (lambda (x) (+ x n))) 2 )
              (lambda (x) (+ x 2))

(  (incrementby 2)    4)
```

Returning procedures

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(define incrementby
  ; type: num -> (num->num)
  (lambda (n) (lambda (x) (+ x n))))

(  incrementby                        2 )
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              (lambda (x) (+ x 2))

(  (incrementby 2)    4)
((lambda (x) (+ x 2)) 4)
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Returning procedures

```
(define incrementby
  ; type: num -> (num->num)
  (lambda (n) (lambda (x) (+ x n))))

(  incrementby                          2 )
( (lambda (n) (lambda (x) (+ x n))) 2 )
              (lambda (x) (+ x 2))

(  (incrementby 2)     4)
((lambda (x) (+ x 2)) 4)
             (+ 4 2)
```

Returning procedures

```
(define incrementby
  ; type: num -> (num->num)
  (lambda (n) (lambda (x) (+ x n))))

(  incrementby                            2 )
( (lambda (n) (lambda (x) (+ x n))) 2 )
               (lambda (x) (+ x 2))

(  (incrementby 2)     4)
((lambda (x) (+ x 2)) 4)
              (+ 4 2)
              6
```

Procedural abstraction

```
(define sqrt (lambda (x) (try 1 x))
```

Procedural abstraction

```
(define sqrt (lambda (x) (try 1 x)))
(define try (lambda (guess x)
              (if (good-enough? guess x)
                  guess
                  (try (improve guess x) x))))
```

Procedural abstraction

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(define sqrt (lambda (x) (try 1 x))
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                       (< (abs (- (square guess)
                                  x))
                          0.001)))
```

Procedural abstraction

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(define improve (lambda (guess x)
                  (average guess (/ x guess))))
```

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(define average (lambda (a b)
                  (/ (+ a b) 2)))
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(define sqrt (lambda (x)
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                  (if (good-enough? guess x)
                      guess
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                           (< (abs (- (square guess)
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                              0.001)))
    (define improve (lambda (guess x)
                      (average guess (/ x guess))))
    (try 1 x))

(define average (lambda (a b)
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Procedural abstraction

```
(define sqrt (lambda (x)
    (define try (lambda (guess  )
                  (if (good-enough? guess  )
                      guess
                      (try (improve guess  )  ))))
    (define good-enough? (lambda (guess  )
                           (< (abs (- (square guess)
                                      x))
                              0.001)))
    (define improve (lambda (guess  )
                      (average guess (/ x guess))))
    (try 1  ))

(define average (lambda (a b)
                  (/ (+ a b) 2)))
```

Summary of types

- A type is a set of values

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- Every value has a type

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Summary of types

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- They provide a mathematical theory for reasoning efficiently about programs
- Useful for preventing some common types of errors
- Basis for many analysis and optimization algorithms

Compound data

- Need a way of (procedure for) gluing data elements together into a unit that can be treated as a simple data element

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Compound data

- Need a way of (procedure for) gluing data elements together into a unit that can be treated as a simple data element
- Need ways of (procedures for) getting the pieces back out
- Need a contract between “glue” and “unglue”
- Ideally want this “gluing” to have the property of closure:
  “The result obtained by creating a compound data structure can itself be treated as a primitive object and thus be input to the creation of another compound object.”

Pairs (`cons` cells)

- `(cons <a> <b>)` → `<p>`

Pairs (cons cells)

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- Where <a> and <b> are expressions that map to <a-val> and <b-val>

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- Returns a pair `<p>` whose car-part is `<a-val>` and whose cdr-part is `<b-val>`
- `(car <p>)` → `<a-val>`
- `(cdr <p>)` → `<b-val>`

Pairs are tasty

```
(define p1 (cons 4 (+ 3 2)))
```

Pairs are tasty

```
(define p1 (cons 4 (+ 3 2)))

(car p1)    ; ->
```

Pairs are tasty

```
(define p1 (cons 4 (+ 3 2)))

(car p1)    ; ->  4
```

Pairs are tasty

```
(define p1 (cons 4 (+ 3 2)))

(car p1)    ; ->  4

(cdr p1)    ; ->
```

Pairs are tasty

```
(define p1 (cons 4 (+ 3 2)))

(car p1)    ; ->  4

(cdr p1)    ; ->  5
```

Pairs are a data abstraction

- Constructor

  `(cons A B)` $\mapsto$ `Pair<A,B>`

Pairs are a data abstraction

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- Contract

  `(car (cons A B))` $\mapsto$ `A`

  `(cdr (cons A B))` $\mapsto$ `B`
- Operations

  `(pair? Q)` returns `#t` if `Q` evaluates to a pair, `#f` otherwise

Pair abstraction

- Once we build a pair, we can treat it as if it were a primitive
- Pairs have the property of closure — we can use a pair anywhere we would expect to use a primitive data element:

```
(cons (cons 1 2) 3)
```

Building data abstractions

```
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
(define (point-y point) (cdr point))
```

Building data abstractions

```
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
(define (point-y point) (cdr point))

(define p1 (make-point 2 3))
```

Building data abstractions

```
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
(define (point-y point) (cdr point))

(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
```

Building data abstractions

```
(define (make-point x y) (cons x y))
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(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
```

What type is `make-point`?

Building data abstractions

```
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
(define (point-y point) (cdr point))

(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
```

What type is `make-point`?

number, number $\mapsto$ Point

Building data abstractions

```
(define make-point cons)
(define point-x car)
(define point-y cdr)

(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
```

Building on earlier abstraction

```
;;; Point abstraction
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
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Building on earlier abstraction

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(define (make-point x y) (cons x y))
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```

Building on earlier abstraction

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;;; Point abstraction
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
(define (point-y point) (cdr point))
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))

;;; Segment abstraction
(define (make-seg pt1 pt2)
  (cons pt1 pt2))
(define (start-point seg)
  (car seg))
(define (end-point seg)
  (cdr seg))
```

Building on earlier abstraction

```
;;; Point abstraction
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
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(define p1 (make-point 2 3))
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;;; Segment abstraction
(define (make-seg pt1 pt2)
  (cons pt1 pt2))
(define (start-point seg)
  (car seg))
(define (end-point seg)
  (cdr seg))
(define s1 (make-seg p1 p2))
```

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
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```

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
(define s1 (make-seg p1 p2))

(define (stretch-point pt scale)
  (make-point (* scale (point-x pt))
              (* scale (point-y pt))))
```

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
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(stretch-point p1 2)  ->
```

Using data abstractions

```
(define p1 (make-point 2 3))
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(define s1 (make-seg p1 p2))

(define (stretch-point pt scale)
  (make-point (* scale (point-x pt))
              (* scale (point-y pt))))

(stretch-point p1 2)  -> (4 . 6)
```

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
(define s1 (make-seg p1 p2))

(define (stretch-point pt scale)
  (make-point (* scale (point-x pt))
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(stretch-point p1 2)  -> (4 . 6)
p1 ->
```

Using data abstractions

```
(define p1 (make-point 2 3))
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(stretch-point p1 2)  -> (4 . 6)
p1 -> (2 . 3)
```

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
(define s1 (make-seg p1 p2))

(define (stretch-point pt scale)
  (make-point (* scale (point-x pt))
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```

What type is `stretch-point`?

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
(define s1 (make-seg p1 p2))

(define (stretch-point pt scale)
  (make-point (* scale (point-x pt))
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```

What type is `stretch-point`?

Point, number $\mapsto$ Point

Using data abstractions

```
(define p1 (make-point 2 3))
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```

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
(define s1 (make-seg p1 p2))

(define (stretch-seg seg scale)
  (make-seg (stretch-point (start-point seg) scale)
            (stretch-point (end-point seg) scale)))
(define (seg-length seg)
  (sqrt (+ (square
             (- (point-x (start-point seg))
                (point-x (end-point seg))))
           (square
             (- (point-y (start-point seg))
                (point-y (end-point seg)))))))
```

Using data abstractions

```
(define p1 (make-point 2 3))
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(define s1 (make-seg p1 p2))

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  (make-seg (stretch-point (start-point seg) scale)
            (stretch-point (end-point seg) scale)))
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  (sqrt (+ (square
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               (point-x (end-point seg))))
           (square
            (- (point-y (start-point seg))
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Using data abstractions

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  (make-seg (stretch-point (start-point seg) scale)
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Using data abstractions

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(stretch-point p1 2)  -> (4 . 6)
p1 -> (2 . 3)
```

Using data abstractions

```
(define p1 (make-point 2 3))
(define p2 (make-point 4 1))
(define s1 (make-seg p1 p2))

(define (stretch-point pt scale)
  (cons       (* scale (car pt))
              (* scale (cdr pt))))

(stretch-point p1 2)  -> (4 . 6)
p1 -> (2 . 3)
```

Abstractions have two communities

- Builders

```
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
```

- Users

```
(* scale (point-x pt))
```

Abstractions have two communities

- Builders
```
(define (make-point x y) (cons x y))
(define (point-x point) (car point))
```
- Users
```
(* scale (point-x pt))
```
- Frequently the same person

Pairs are a data abstraction

- Constructor

  `(cons A B)` $\mapsto$ `Pair<A,B>`
- Accessors

  `(car Pair<A,B>)` $\mapsto$ `A`

  `(cdr Pair<A,B>)` $\mapsto$ `B`
- Contract

  `(car (cons A B))` $\mapsto$ `A`

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  `(car (cons A B))` ↦ `A`

  `(cdr (cons A B))` ↦ `B`

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  `(pair? Q)` returns `#t` if `Q` evaluates to a pair, `#f` otherwise

- Abstraction barrier

Rational number abstraction

- A rational number is a ratio $\frac{n}{d}$

Rational number abstraction

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- Addition:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

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Rational number abstraction

- A rational number is a ratio $\frac{n}{d}$
- Addition:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

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- Multiplication:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

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- A rational number is a ratio $\frac{n}{d}$
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$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

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- Multiplication:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

Rational number abstraction

- Constructor

```
; make-rat:  integer, integer -> Rat
(make-rat <n> <d>) -> <r>
```

Rational number abstraction

- Constructor

```
; make-rat:  integer, integer -> Rat
(make-rat <n> <d>) -> <r>
```

- Accessors

```
; numer, denom:  Rat -> integer
(numer <r>)
(denom <r>)
```

Rational number abstraction

- Constructor

```
; make-rat:  integer, integer -> Rat
(make-rat <n> <d>) -> <r>
```

- Accessors

```
; numer, denom:  Rat -> integer
(numer <r>)
(denom <r>)
```

- Contract

```
(numer (make-rat <n> <d>)) ⟹ <n>
(denom (make-rat <n> <d>)) ⟹ <d>
```

Rational number abstraction

- Constructor

```
; make-rat:  integer, integer -> Rat
(make-rat <n> <d>) -> <r>
```

- Accessors

```
; numer, denom:  Rat -> integer
(numer <r>)
(denom <r>)
```

- Contract

```
(numer (make-rat <n> <d>)) ⟹ <n>
(denom (make-rat <n> <d>)) ⟹ <d>
```

- Operations

```
(+rat x y)
(*rat x y)
```

Rational number abstraction

- Constructor

```
; make-rat:  integer, integer -> Rat
(make-rat <n> <d>) -> <r>
```

- Accessors

```
; numer, denom:  Rat -> integer
(numer <r>)
(denom <r>)
```

- Contract

```
(numer (make-rat <n> <d>)) ⟹ <n>
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(+rat x y)
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- Abstraction barrier

Rational number abstraction

- Constructor
- Accessors
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- Abstraction barrier
- Implementation

```
; Rat = Pair<integer, integer>
(define (make-rat n d) (cons n d))
(define (numer r) (car r))
(define (denom r) (cdr r))
```

Rational number abstraction

- Constructor
- Accessors
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- Operations
- Abstraction barrier

[redacted]

- Implementation

```
; Rat = Pair<integer, integer>
(define (make-rat n d) (cons d n))
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Additional operators

```
; What is the type of +rat?
(define (+rat x y)
  (make-rat (+ (* (numer x) (denom y))
               (* (numer y) (denom x)))
            (* (denom x) (denom y))))
```

Additional operators

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; What is the type of +rat? Rat, Rat -> Rat
(define (+rat x y)
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; The type of *rat:
(define (*rat x y)
  (make-rat (* (numer x) (numer y))
            (* (denom x) (denom y))))
```

Additional operators

```
; What is the type of +rat? Rat, Rat -> Rat
(define (+rat x y)
  (make-rat (+ (* (numer x) (denom y))
               (* (numer y) (denom x)))
            (* (denom x) (denom y))))

; The type of *rat: Rat, Rat -> Rat
(define (*rat x y)
  (make-rat (* (numer x) (numer y))
            (* (denom x) (denom y))))
```

Using our system

```
(define one-half (make-rat 1 2))
(define three-fourths (make-rat 3 4))

(define new (+rat one-half three-fourths))

(numer new)      ; ?
(denom new)      ; ?
```

Using our system

```
(define one-half (make-rat 1 2))
(define three-fourths (make-rat 3 4))

(define new (+rat one-half three-fourths))

(numer new)     ; 10
(denom new)     ; 8
```

We get $\frac{10}{8}$, not the simplified $\frac{5}{4}$

Rationalizing implementation

```
(define (gcd a b)
  (if (= b 0)
      a
      (gcd b (remainder a b))))
```

Rationalizing implementation

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(define (gcd a b)
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(define (make-rat n d)
  (cons n d))

(define (numer r)
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(define (denom r)
  (/ (cdr r) (gcd (car r) (cdr r))))
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(define (denom r)
  (/ (cdr r) (gcd (car r) (cdr r))))
```

Remove common factors when accessed

Rationalizing implementation

```
(define (gcd a b)
  (if (= b 0)
      a
      (gcd b (remainder a b))))

(define (make-rat n d)
  (cons (/ n (gcd n d))
        (/ d (gcd n d))))
(define (numer r)
  (car r))
(define (denom r)
  (cdr r))
```

Remove common factors when created

Grouping together larger collections

We want to group a set of rational numbers

Grouping together larger collections

We want to group a set of rational numbers

```
(cons r1 r2)
```

Grouping together larger collections

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```
(cons (cons r1 r2)
      (cons r3 r4))
```

Grouping together larger collections

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```
(cons (cons (cons r1 r2)
            (cons r3 r4))
      r5)
```

Grouping together larger collections

We want to group a set of rational numbers

```
(cons (cons (cons r1 r2)
            (cons r3 r4))
      (cons r5 r6))
```

Grouping together larger collections

We want to group a set of rational numbers

```
(cons (cons (cons (cons r1 r2)
                  (cons r3 r4))
            (cons r5 r6))
      (cons r7 r8))
```

Grouping together larger collections

We want to group a set of rational numbers

```
(cons (cons (cons r1 r2)
            (cons r3 r4))
      (cons (cons r5 r6)
            (cons r7 r8)))
```

Grouping together larger collections

We want to group a set of rational numbers

```
(cons (cons (cons r1 r2)
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...

Conventional interfaces — lists

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- Formally, a list is a sequence of pairs with the following properties:
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  - The list is terminated by the empty list: `'()`
- Lists are closed under `cons` and `cdr`

Lists and pairs as pictures

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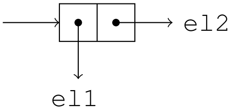
```



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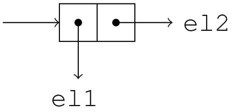


```
(list <el1> <el2> ... <eln>)
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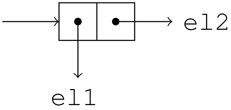


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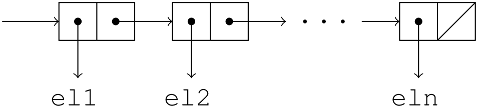

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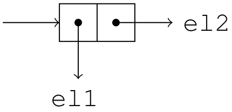


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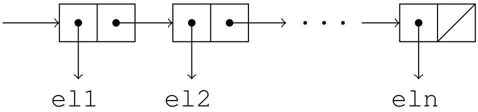


```
(list 1 2 3 4) ; ->
```

Lists and pairs as pictures

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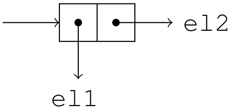
(cons <el1> <el2>)
```

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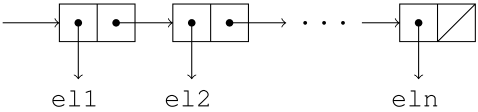


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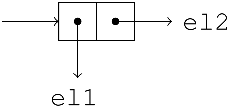


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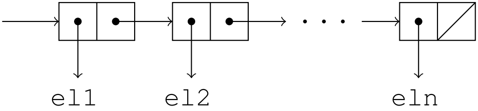


```
(list 1 2 3 4) ; -> (1 2 3 4)
(null? <z>) ; -> #t if <z> evaluates to empty list
```

Lists

- Sequences of `cons` cells
- Better, and safer, to abstract:

```
(define first car)
(define rest cdr)
(define adjoin cons)
```

Lists

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- Better, and safer, to abstract:

```
(define first car)
(define rest cdr)
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```

- ... but we don't for lists and pairs

cons'ing up lists

```
(define 1thru4 (list 1 2 3 4))
```

cons'ing up lists

```
(define 1thru4 (list 1 2 3 4))
(define 2thru7 (list 2 3 4 5 6 7))
```

cons'ing up lists

```
(define 1thru4 (list 1 2 3 4))
(define 2thru7 (list 2 3 4 5 6 7))

(define (enumerate from to)
  (if (> from to)
      '()
      (cons from (enumerate (+ 1 from) to))))
```

cdr'ing down lists

```
(define (length lst)
  (if (null? lst)
      0
      (+ 1 (length (cdr lst)))))
```

`cdr`'ing down lists

```
(define (length lst)
  (if (null? lst)
      0
      (+ 1 (length (cdr lst)))))

(define (append list1 list2)
  (if (null? list1)
      list2
      (cons (car list1)
            (append (cdr list1)
                    list2))))
```

Transforming lists

```
(define (square-list lst)
  (if (null? lst)
      '()
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(define (map proc lst)
  (if (null? lst)
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Map

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(define (map proc lst)
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What is the type of `map`?

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What is the type of `map`?

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Choosing just part of a list

```
(define (filter pred lst)
  (cond ((null? lst) '())
        ((pred (car lst))
         (cons (car lst)
               (filter pred (cdr lst))))
        (else (filter pred (cdr lst)))))

(filter even? (list 1 2 3 4 5 6))
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  - Pairs and lists: `(42 . 8)`, `(1 1 2 3 5 8 13)`
  - Symbols: `pi`, `+`, `x`, `foo`, `hello-world`

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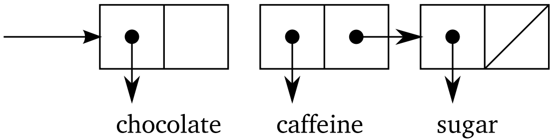
  - Associate them as values of variables
  - Store them in data structures
    - For example: `(chocolate caffeine sugar)`



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- In the first case, we want the meaning associated with the expression
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- We use the concept of quotation in Scheme to distinguish between these two cases

New special form: quote

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  → `(foo bar baz)`

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  `'pi` → `pi`

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Making list structures with symbols

```
(list (quote brains) (quote caffeine) (quote sugar))
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      ; -> (brains caffeine sugar)
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(list (quote brains) (quote caffeine) (quote sugar))
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(list 'brains 'caffeine 'sugar)
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(list (quote brains) (quote caffeine) (quote sugar))
       ; -> (brains caffeine sugar)
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       ; -> (brains caffeine sugar)
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```

Making list structures with symbols

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(list (quote brains) (quote caffeine) (quote sugar))
      ; -> (brains caffeine sugar)
(list 'brains 'caffeine 'sugar)
      ; -> (brains caffeine sugar)
'(brains caffeine sugar)
      ; -> (brains caffeine sugar)
```

Making list structures with symbols

```
(list (quote brains) (quote caffeine) (quote sugar))
      ; -> (brains caffeine sugar)
(list 'brains 'caffeine 'sugar)
      ; -> (brains caffeine sugar)
'(brains caffeine sugar)
      ; -> (brains caffeine sugar)
(define x 42) (define y '(x y z))
```

Making list structures with symbols

```
(list (quote brains) (quote caffeine) (quote sugar))
      ; -> (brains caffeine sugar)
(list 'brains 'caffeine 'sugar)
      ; -> (brains caffeine sugar)
'(brains caffeine sugar)
      ; -> (brains caffeine sugar)
(define x 42) (define y '(x y z))
(list  (list 'foo 'bar) (list x y)
       (list 'baz 'quux 'squee))
```

Making list structures with symbols

```
(list (quote brains) (quote caffeine) (quote sugar))
      ; -> (brains caffeine sugar)
(list 'brains 'caffeine 'sugar)
      ; -> (brains caffeine sugar)
'(brains caffeine sugar)
      ; -> (brains caffeine sugar)
(define x 42) (define y '(x y z))
(list  (list 'foo 'bar) (list x y)
       (list 'baz 'quux 'squee))
      ; -> ((foo bar) (42 (x y z))
           (baz quux squee))
```

Making list structures with symbols

```
(list (quote brains) (quote caffeine) (quote sugar))
       ; -> (brains caffeine sugar)
(list 'brains 'caffeine 'sugar)
       ; -> (brains caffeine sugar)
'(brains caffeine sugar)
       ; -> (brains caffeine sugar)
(define x 42) (define y '(x y z))
(list  (list 'foo 'bar) (list x y)
       (list 'baz 'quux 'squee))
       ; -> ((foo bar) (42 (x y z))
             (baz quux squee))
'((foo bar) (x y) (bar quux squee))
```

Making list structures with symbols

```
(list (quote brains) (quote caffeine) (quote sugar))
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(list 'brains 'caffeine 'sugar)
        ; -> (brains caffeine sugar)
'(brains caffeine sugar)
        ; -> (brains caffeine sugar)
(define x 42) (define y '(x y z))
(list  (list 'foo 'bar) (list x y)
        (list 'baz 'quux 'squee))
        ; -> ((foo bar) (42 (x y z))
               (baz quux squee))
'((foo bar) (x y) (bar quux squee))
        ; -> ((foo bar) (x y) (bar quux squee))
```

Confusing examples

```
(define x 20)
```

Confusing examples

```
(define x 20)
(+ x 3)                     ; ->
```

Confusing examples

```
(define x 20)
(+ x 3)                    ; ->  23
```

Confusing examples

```
(define x 20)
(+ x 3)                 ; ->  23
'(+ x 3)                ; ->
```

Confusing examples

```
(define x 20)
(+ x 3)                    ; ->  23
'(+ x 3)                   ; ->  (+ x 3)
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(define x 20)
(+ x 3)                  ; ->  23
'(+ x 3)                 ; ->  (+ x 3)
(list (quote +) x '3)    ; ->
```

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'(+ x 3)                ; ->  (+ x 3)
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Confusing examples

```
(define x 20)
(+ x 3)                   ; ->  23
'(+ x 3)                  ; ->  (+ x 3)
(list (quote +) x '3)     ; ->  (+ 20 3)
(list '+ x 3)             ; ->
```

Confusing examples

```
(define x 20)
(+ x 3)                 ; ->  23
'(+ x 3)                ; ->  (+ x 3)
(list (quote +) x '3)   ; ->  (+ 20 3)
(list '+ x 3)           ; ->  (+ 20 3)
```

Confusing examples

```
(define x 20)
(+ x 3)                    ; ->  23
'(+ x 3)                   ; ->  (+ x 3)
(list (quote +) x '3)      ; ->  (+ 20 3)
(list '+ x 3)              ; ->  (+ 20 3)
(list + x 3)               ; ->
```

Confusing examples

```
(define x 20)
(+ x 3)                  ; ->  23
'(+ x 3)                 ; ->  (+ x 3)
(list (quote +) x '3)    ; ->  (+ 20 3)
(list '+ x 3)            ; ->  (+ 20 3)
(list + x 3)             ; ->  (#<procedure:+> 20 3)
```

Operations on symbols

- `symbol?` has type `anytype` $\rightarrow$ `boolean`, returns `#t` for symbols

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  `(symbol? 4)` → `#f`

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  ```
  (symbol? (quote foo)) → #t
  (symbol? 'foo) → #t
  (symbol? 4) → #f
  (symbol? '(1 2 3)) → #f
  ```

Operations on symbols

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  `(symbol? (quote foo))` $\rightarrow$ `#t`

  `(symbol? 'foo)` $\rightarrow$ `#t`

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  `(symbol? '(1 2 3))` $\rightarrow$ `#f`

  `(symbol? foo)` $\rightarrow$ It depends on what value `foo` is bound to

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  `(symbol? 'foo)` → `#t`

  `(symbol? 4)` → `#f`

  `(symbol? '(1 2 3))` → `#f`

  `(symbol? foo)` → It depends on what value `foo` is bound to
- `eq?` tests the equality of symbols

An aside: Testing for equality

- `eq?` tests if two things are exactly the same object in memory. Not for strings or numbers.

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An aside: Testing for equality

- `eq?` tests if two things are exactly the same object in memory. Not for strings or numbers.
- = tests the equality of numbers
- `equal?` tests if two things print the same– symbols, numbers, strings, lists of those, lists of lists

```
(= 4 10)                    ; ->
```

```
(= 4 10)                              ; ->  #f
```

```
(= 4 10)                    ; -> #f
(= 4 4)                     ; ->
```

```
(= 4 10)                      ; -> #f
(= 4 4)                       ; -> #t
```

```
(= 4 10)          ; -> #f
(= 4 4)           ; -> #t
(equal? 4 4)      ; ->
```

```
(= 4 10)          ; -> #f
(= 4 4)           ; -> #t
(equal? 4 4)      ; -> #t
```

```scheme
(= 4 10)                    ; ->  #f
(= 4 4)                     ; ->  #t
(equal? 4 4)                ; ->  #t
(equal? (/ 1 2) 0.5)        ; ->
```

```scheme
(= 4 10)                  ; ->  #f
(= 4 4)                   ; ->  #t
(equal? 4 4)              ; ->  #t
(equal? (/ 1 2) 0.5)      ; ->  #f
```

```
(= 4 10)                        ; ->  #f
(= 4 4)                         ; ->  #t
(equal? 4 4)                    ; ->  #t
(equal? (/ 1 2) 0.5)            ; ->  #f
(eq? 4 4)                       ; ->
```

```
(= 4 10)                      ; ->  #f
(= 4 4)                       ; ->  #t
(equal? 4 4)                  ; ->  #t
(equal? (/ 1 2) 0.5)          ; ->  #f
(eq? 4 4)                     ; ->  #t
```

```
(= 4 10)                        ; ->  #f
(= 4 4)                         ; ->  #t
(equal? 4 4)                    ; ->  #t
(equal? (/ 1 2) 0.5)            ; ->  #f
(eq? 4 4)                       ; ->  #t
(eq? (expt 2 70) (expt 2 70)) ; ->
```

```scheme
(= 4 10)                          ; ->  #f
(= 4 4)                           ; ->  #t
(equal? 4 4)                      ; ->  #t
(equal? (/ 1 2) 0.5)              ; ->  #f
(eq? 4 4)                         ; ->  #t
(eq? (expt 2 70) (expt 2 70))     ; ->  #f
```

```scheme
(= 4 10)                          ; ->  #f
(= 4 4)                           ; ->  #t
(equal? 4 4)                      ; ->  #t
(equal? (/ 1 2) 0.5)              ; ->  #f
(eq? 4 4)                         ; ->  #t
(eq? (expt 2 70) (expt 2 70))     ; ->  #f

(= "foo" "foo")                   ; ->
```

```scheme
(= 4 10)                        ; ->  #f
(= 4 4)                         ; ->  #t
(equal? 4 4)                    ; ->  #t
(equal? (/ 1 2) 0.5)            ; ->  #f
(eq? 4 4)                       ; ->  #t
(eq? (expt 2 70) (expt 2 70))   ; ->  #f

(= "foo" "foo")                 ; ->  Error!
```

```
(= 4 10)                         ; ->  #f
(= 4 4)                          ; ->  #t
(equal? 4 4)                     ; ->  #t
(equal? (/ 1 2) 0.5)             ; ->  #f
(eq? 4 4)                        ; ->  #t
(eq? (expt 2 70) (expt 2 70))    ; ->  #f

(= "foo" "foo")                  ; ->  Error!
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(= 4 10)                          ; -> #f
(= 4 4)                           ; -> #t
(equal? 4 4)                      ; -> #t
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```scheme
(= 4 10)                       ; -> #f
(= 4 4)                        ; -> #t
(equal? 4 4)                   ; -> #t
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(eq? 4 4)                      ; -> #t
(eq? (expt 2 70) (expt 2 70))  ; -> #f

(= "foo" "foo")                ; -> Error!
(eq? "foo" "foo")              ; -> #f
(equal? "foo" "foo")           ; ->
```

```scheme
(= 4 10)                           ; -> #f
(= 4 4)                            ; -> #t
(equal? 4 4)                       ; -> #t
(equal? (/ 1 2) 0.5)               ; -> #f
(eq? 4 4)                          ; -> #t
(eq? (expt 2 70) (expt 2 70))      ; -> #f

(= "foo" "foo")                    ; -> Error!
(eq? "foo" "foo")                  ; -> #f
(equal? "foo" "foo")               ; -> #t
```

```
(= 4 10)                        ; ->  #f
(= 4 4)                         ; ->  #t
(equal? 4 4)                    ; ->  #t
(equal? (/ 1 2) 0.5)            ; ->  #f
(eq? 4 4)                       ; ->  #t
(eq? (expt 2 70) (expt 2 70))   ; ->  #f

(= "foo" "foo")                 ; ->  Error!
(eq? "foo" "foo")               ; ->  #f
(equal? "foo" "foo")            ; ->  #t

(eq? '(1 2) '(1 2))             ; ->
```

```scheme
(= 4 10)                          ; ->  #f
(= 4 4)                           ; ->  #t
(equal? 4 4)                      ; ->  #t
(equal? (/ 1 2) 0.5)              ; ->  #f
(eq? 4 4)                         ; ->  #t
(eq? (expt 2 70) (expt 2 70))     ; ->  #f

(= "foo" "foo")                   ; ->  Error!
(eq? "foo" "foo")                 ; ->  #f
(equal? "foo" "foo")              ; ->  #t

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```

```
(= 4 10)                          ; ->  #f
(= 4 4)                           ; ->  #t
(equal? 4 4)                      ; ->  #t
(equal? (/ 1 2) 0.5)              ; ->  #f
(eq? 4 4)                         ; ->  #t
(eq? (expt 2 70) (expt 2 70))     ; ->  #f

(= "foo" "foo")                   ; ->  Error!
(eq? "foo" "foo")                 ; ->  #f
(equal? "foo" "foo")              ; ->  #t

(eq? '(1 2) '(1 2))               ; ->  #f
(equal? '(1 2) '(1 2))            ; ->
```

```
(= 4 10)                          ; ->  #f
(= 4 4)                           ; ->  #t
(equal? 4 4)                      ; ->  #t
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(eq? 4 4)                         ; ->  #t
(eq? (expt 2 70) (expt 2 70))     ; ->  #f

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(equal? "foo" "foo")              ; ->  #t

(eq? '(1 2) '(1 2))               ; ->  #f
(equal? '(1 2) '(1 2))            ; ->  #t
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(= 4 10)                          ; ->  #f
(= 4 4)                           ; ->  #t
(equal? 4 4)                      ; ->  #t
(equal? (/ 1 2) 0.5)              ; ->  #f
(eq? 4 4)                         ; ->  #t
(eq? (expt 2 70) (expt 2 70))     ; ->  #f

(= "foo" "foo")                   ; ->  Error!
(eq? "foo" "foo")                 ; ->  #f
(equal? "foo" "foo")              ; ->  #t

(eq? '(1 2) '(1 2))               ; ->  #f
(equal? '(1 2) '(1 2))            ; ->  #t
(define a '(1 2))
```

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(= 4 10)                          ; ->  #f
(= 4 4)                           ; ->  #t
(equal? 4 4)                      ; ->  #t
(equal? (/ 1 2) 0.5)              ; ->  #f
(eq? 4 4)                         ; ->  #t
(eq? (expt 2 70) (expt 2 70))     ; ->  #f

(= "foo" "foo")                   ; ->  Error!
(eq? "foo" "foo")                 ; ->  #f
(equal? "foo" "foo")              ; ->  #t

(eq? '(1 2) '(1 2))               ; ->  #f
(equal? '(1 2) '(1 2))            ; ->  #t
(define a '(1 2))
(define b '(1 2))
```

```
(= 4 10)                        ; ->  #f
(= 4 4)                         ; ->  #t
(equal? 4 4)                    ; ->  #t
(equal? (/ 1 2) 0.5)            ; ->  #f
(eq? 4 4)                       ; ->  #t
(eq? (expt 2 70) (expt 2 70))   ; ->  #f

(= "foo" "foo")                 ; ->  Error!
(eq? "foo" "foo")               ; ->  #f
(equal? "foo" "foo")            ; ->  #t

(eq? '(1 2) '(1 2))             ; ->  #f
(equal? '(1 2) '(1 2))          ; ->  #t
(define a '(1 2))
(define b '(1 2))
(eq? a b)               ; ->
```

```scheme
(= 4 10)                          ; -> #f
(= 4 4)                           ; -> #t
(equal? 4 4)                      ; -> #t
(equal? (/ 1 2) 0.5)              ; -> #f
(eq? 4 4)                         ; -> #t
(eq? (expt 2 70) (expt 2 70))     ; -> #f

(= "foo" "foo")                   ; -> Error!
(eq? "foo" "foo")                 ; -> #f
(equal? "foo" "foo")              ; -> #t

(eq? '(1 2) '(1 2))               ; -> #f
(equal? '(1 2) '(1 2))            ; -> #t
(define a '(1 2))
(define b '(1 2))
(eq? a b)                  ; -> #f
```

```scheme
(= 4 10)                         ; -> #f
(= 4 4)                          ; -> #t
(equal? 4 4)                     ; -> #t
(equal? (/ 1 2) 0.5)             ; -> #f
(eq? 4 4)                        ; -> #t
(eq? (expt 2 70) (expt 2 70))    ; -> #f

(= "foo" "foo")                  ; -> Error!
(eq? "foo" "foo")                ; -> #f
(equal? "foo" "foo")             ; -> #t

(eq? '(1 2) '(1 2))              ; -> #f
(equal? '(1 2) '(1 2))           ; -> #t
(define a '(1 2))
(define b '(1 2))
(eq? a b)                 ; -> #f
(define a b)
```

```
(= 4 10)                        ; -> #f
(= 4 4)                         ; -> #t
(equal? 4 4)                    ; -> #t
(equal? (/ 1 2) 0.5)            ; -> #f
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(= "foo" "foo")                 ; -> Error!
(eq? "foo" "foo")               ; -> #f
(equal? "foo" "foo")            ; -> #t

(eq? '(1 2) '(1 2))             ; -> #f
(equal? '(1 2) '(1 2))          ; -> #t
(define a '(1 2))
(define b '(1 2))
(eq? a b)                 ; -> #f
(define a b)
(eq? a b)                 ; ->
```

```
(= 4 10)                        ; ->  #f
(= 4 4)                         ; ->  #t
(equal? 4 4)                    ; ->  #t
(equal? (/ 1 2) 0.5)            ; ->  #f
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(equal? "foo" "foo")            ; ->  #t

(eq? '(1 2) '(1 2))             ; ->  #f
(equal? '(1 2) '(1 2))          ; ->  #t
(define a '(1 2))
(define b '(1 2))
(eq? a b)               ; ->  #f
(define a b)
(eq? a b)               ; ->  #t
```

Tagged data

- Attaching a symbol to all data values that indicates the type
- Can now determine if something is the type you expect

```
(define (make-point x y)
  (list x y))

(define (make-rat n d)
  (list x y))
```

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- Attaching a symbol to all data values that indicates the type
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(define (make-point x y)
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- Attaching a symbol to all data values that indicates the type
- Can now determine if something is the type you expect

```
(define (make-point x y)
  (list 'point x y))

(define (make-rat n d)
  (list 'rat x y))

(define (point? thing)
  (and (pair? thing)
       (eq? (car thing) 'point)))

(define (rat? thing)
  (and (pair? thing)
       (eq? (car thing) 'rat)))
```

Benefits of tagged data

- Data-directed programming - decide what to do based on type

```
(define (stretch thing scale)
    (if (point? thing)
        (stretch-point thing scale)
        (stretch-seg   thing scale)))
```

Benefits of tagged data

- Data-directed programming - decide what to do based on type

```
(define (stretch thing scale)
    (if (point? thing)
        (stretch-point thing scale)
        (stretch-seg   thing scale)))
```

- Defensive programming - Determine if something is the type you expect, give a better error

```
(define (stretch-point pt)
  (if (not (point? pt))
      (error "stretch-point passed a non-point:" pt)
      ;; ...carry on
  ))
```

Recitation time!