# CSC 411: Lecture 16: Kernels

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# Today

- Kernel trick

# Summary of Linear SVM

- Binary and linear separable classification

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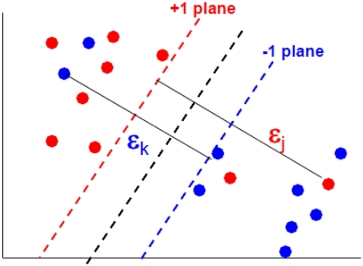
- Prediction on a new example:

$$y = \text{sign}[b + \mathbf{x} \cdot (\sum_{i=1}^{N} \alpha_i t^{(i)} \mathbf{x}^{(i)})] = \text{sign}[b + \mathbf{x} \cdot (\sum_{i \in \mathbf{S}} \alpha_i t^{(i)} \mathbf{x}^{(i)})]$$

# What if data is not linearly separable?




- Introduce slack variables $\xi_i$

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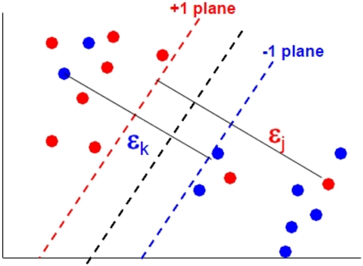


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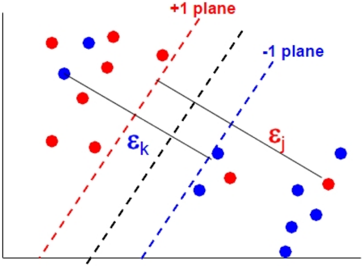


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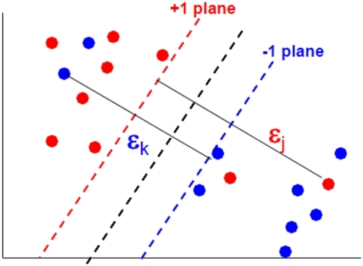


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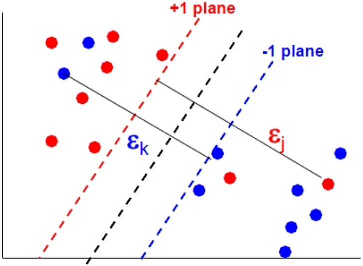


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- This is known as the soft-margin extension

# Non-linear Decision Boundaries

- Note that both the learning objective and the decision function depend only on dot products between patterns

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- Problem: what is a good feature function $\phi(\mathbf{x})$?

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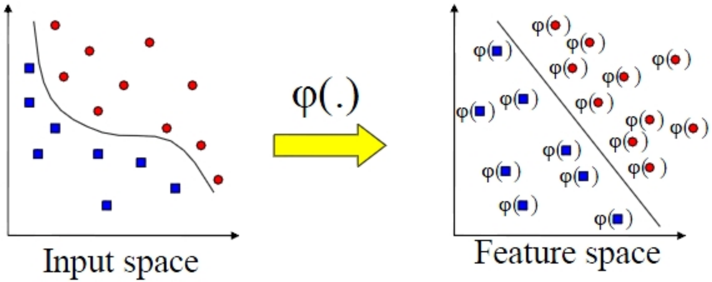


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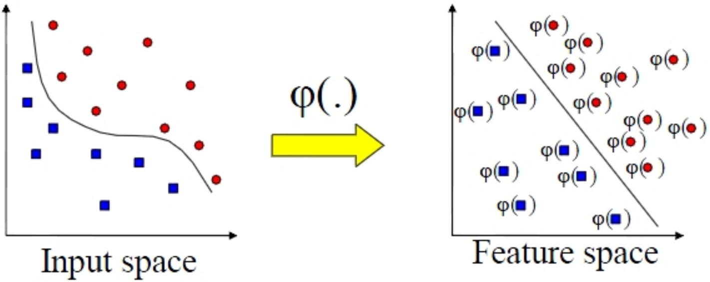


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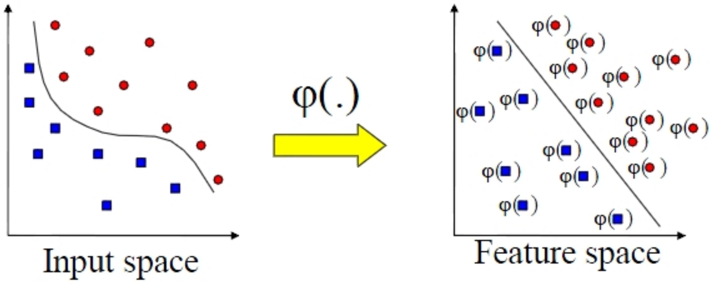


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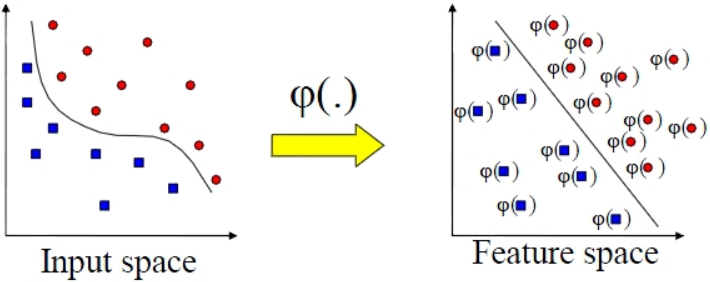


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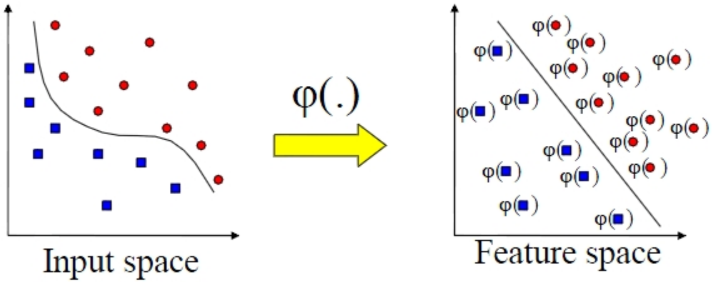


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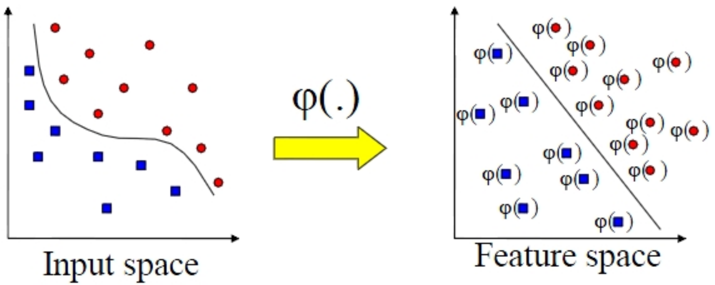


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  - Dual formulation only assigns parameters to samples, not features

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  1. Rewrite training examples using more complex features
  2. Dataset not linearly separable in original space may be linearly separable in higher dimensional space

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  - Gaussian kernel has infinitely dimensional features
- Linear separators in these super high-dim spaces correspond to highly nonlinear decision boundaries in input space

# Classification with Non-linear SVMs

- Non-linear SVM using kernel function $K()$:

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